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THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

The Association of Mathematics Teachers of India (AMTI) was started in 1965 for the promotion of efforts to improve Mathematics education at all levels. A major aim of the Association is to assist practicing teachers of Mathematics in schools in improving their expertise and professional skills. Another important aim is to spot out and foster Mathematical talent in the young. The Association also seeks to disseminate new trends in Mathematics education among parents and public. Other activities of the Association include consultancy services to schools in equipping the Mathematics section of their libraries, in organising children's Mathematics clubs and fairs, in setting up teacher centers in schools, in conducting Mathematics laboratory programmes, in holding practical tests in Mathematics in assisting children in participating investigational projects.

The Association holds "Inter State Mathematical Talent Search Competition" annually and organises Orientation Courses, Seminars and Workshops for teachers and summer courses for talented students. A national conference is held annually in different parts of the country for teachers to meet and deliberate on important issues of Mathematics education. Innovative teacher award has been instituted to give public recognition to enterprising and pioneering teachers of Mathematics for which entries from teachers are invited.

An award for contributions to the Mathematics Teacher relating to History of Mathematics in the context of mathematics education has been instituted by Prof. R.C.Gupta.

"The Mathematics Teacher (India)" (MT) is the official journal of the Association and is published quarterly. It has been approved for use in schools and colleges of education by the Government departments of education in many States. Besides MT the Association also brings out Junior Mathematician (JM), three issues in a year, especially for school students in three languages – English, Hindi and Tamil.

The membership of the Association is open to all those interested in Mathematics and Mathematics Education. The membership fee inclusive of subscription for the MT and effective from April 1993:

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Life	Rs. 500	Rs. 1000
Annual (Ordinary)	Rs. 50	---
Junior Mathematician – Life	Rs. 250	Rs. 250
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The Journal "The Mathematics Teacher" will be supplied free to all members except student members. Student members are eligible for supply of Junior Mathematician only. If student members need Mathematics Teacher, they have to pay Rs. 30 more and become regular members also.

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i.e instead of rupees read US dollars

The Mathematics Teacher

(INDIA)

OFFICIAL JOURNAL OF THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

Volume: 38 Issues: 1 & 2 Year: 2002

Prof. A.Ramakrishnan

Editor

EDITORIAL

In this combined first and second issues of Volume 38, the solutions to the problems of the Screening and Final tests of AMTI Talent Competitions 2001 are given. Besides the solutions to the problems of Regional Mathematics Olympiad and the Indian Mathematics Olympiad also are included. The editor is extremely grateful to Prof. V.K.Krishnan, Dr. V.Shankaram, Prof. K.N.Ranganathan, Ms. R.Vijayalakshmi, Dr.(Mrs) Hemalatha Thiagarajan and Mr. R.Athmaraman for the valuable support given by them in this task.

The contributions to the present day Mathematics by the Indians is being felt world over. Two articles regarding the valuable contributions of IIT, Kanpur appear in this volume. We express our grateful thanks to Shri S.Ananthanarayanan and K.Venkatesh.

Prof. K.S.Padmanabhan, former Director and Head, Ramanujan Institute for Advanced Study in Mathematics, University of Madras passed away on 2-6-2002 in Chennai. A brief sketch of his life and works is given in this issue. He communicated to the Editor of Mathematics Teacher, a generalization of a problem given in Mathematics Teacher Vol.35 and this, perhaps his last contribution to a Mathematics journal, is published in this issue.

Students of Mathematics have always found that they have been influenced by their Mathematics teacher sometime or other in their career. Some of these teachers have been very outstanding. The editor of Mathematics Teacher welcomes contributions from the readers under the head "My Mathematics Teacher" giving their teacher's main contributions to the teaching of Mathematics and detailing how they created an interest in their students to love and study Mathematics.

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AMTI

SCREENING TEST - JUNIOR LEVEL - 2001

Note:

1. *Attempt all questions*
2. *For each question choose the most appropriate answer*
3. *Write the answer in the response sheet supplied separately*
4. *You may use a rough sheet, if necessary, to do calculation*
5. *Calculator/Log tables are not to be used*
6. *In the case of doubts regarding the mode of answering ask your supervisor for clarification*
7. THE TIME ALLOTTED FOR ANSWERING IS 2 HOURS ONLY
8. *Figures in the questions are not drawn to scale*

1. If a, b, c are real numbers such that

$$a + \left(\frac{1}{b}\right) = \frac{7}{3}, b + \left(\frac{1}{c}\right) = 4, c + \left(\frac{1}{a}\right) = 1 \text{ then } abc =$$

(A) 2 (B) 1 (C) $\frac{2}{3}$ (D) $\frac{2}{3}$

Solution:

$$\frac{28}{3} = \frac{7}{3} \times 4 \times 1$$

$$= (a + \frac{1}{b})(b + \frac{1}{c})(c + \frac{1}{a})$$

$$= abc + (a + b + c) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{1}{abc}$$

$$= abc + \frac{1}{abc} + \frac{22}{3}$$

$\therefore \quad abc + \frac{1}{abc} = 2$

$\therefore \quad (abc - 1)^2 = 0 \quad \Rightarrow \quad abc = 1.$

Answer is (B).

2. If $\{(3(230+x))^2\} = 492a04$ then the value of a is

(A) 4 (B) 3 (C) 8 (D) 2

Solution: LHS is divisible by 9.
$\therefore$ RHS is divisible by 9 (and also $o < a \leq 9$)
4+9+2+a+0+4 (i.e., $19+a$) is divisible by 9.
$\therefore$ $a = 8$.

Answer is (C).

3. If $200 \leq a \leq 400$ and $600 \leq b \leq 1600$, then the largest possible value of the rational number $\frac{b}{a}$ is

(A) 3 (B) 8 (C) $\frac{3}{2}$ (D) $\frac{11}{4}$

Solution: We want $\frac{b}{a}$ to be the largest.
$\therefore$ b should be the greatest and a the least.
$\therefore$ Required value of $\frac{b}{a} = \frac{1600}{200} = 8$.
Answer is (B).

4. If $2^{1998} - 2^{1999} - 2^{2000} + 2^{2001} = k \cdot 2^{2002}$ then the value of k is

(A) $\frac{16}{3}$ (B) $\frac{3}{16}$ (C) $\frac{8}{3}$ (D) $\frac{3}{8}$

Solution: $2^{1998}(1 - 2 - 2^2 + 2^3) = k \cdot 2^{2002}$
$\therefore$ $3 = k \cdot 2^4$ $\therefore$ $k = \frac{3}{16}$ Answer is (D).

5. The number of Pythagorean triples (a, b, c) where $a^2 + b^2 = c^2$ and each integer a, b, c being odd is

(A) 3 (B) 1 (C) 0 (D) infinite

Solution: There cannot be a Pythagorean triple where each of the numbers is odd. For if a, b are odd, then a^2+b^2 is even and c cannot be odd.

$\therefore$ Answer is (C).

6. In the year 2002, Great Britain will host the International Mathematics Olympiad. Let I, M, O be distinct positive integers such that the product $I.M.O = 2002$. Then the the largest possible value of the sum $I + M + O$ is

(A) 1004 (B) 158 (C) 2004 (D) 114

Solution: 1,2,7,286,182,11, 13,154, 1001,2002, are the divisors of 2002.

$\therefore$ the largest $I + M + O = 1 + 2 + 1001 = 1004$.

Answer is (A).

7. At the end of the year 1994, Ram was half as old as his grandfather while the sum of the years in which they were born is 3844. The age of Ram at the end of 2001 is

(A) 20 (B) 21 (C) 55 (D) 57

Solution: In 1994 if age of Ram is x, then his grandfather's age is $2x$. So their births took place in $1994 - x$ and $1994 - 2x$. Thus $3844 = 3988 - 3x$ so that $x = 48$. Hence in 2001 the age of Ram is $x + 7 = 55$. Answer is (C).

8. If a, b, c are real numbers such that
$(2a - 3)^2 + (4b - 5)^2 + (6c - 7)^2 = 0$, then

(A) $abc = a + b + c$ (B) $abc > a + b + c$
(C) $abc < a + b + c$ (D) $abc = \sqrt{a + b + c}$

Solution: Sum of squares $= 0 \Rightarrow$ Each term $= 0$.

$\therefore \quad a = \frac{3}{2}; b = \frac{5}{4}; c = \frac{7}{6}.$

$\therefore \quad a + b + c = \frac{47}{12}$ and $abc = \frac{105}{48}$. We find $abc < a + b + c$.

Answer is (C).

9. If the number $N = 30a0b03$ is divisible by 13, then a cannot take the value.

(A) 3 (B) 2 (C) 1 (D) 0

Solution: $N = 3000003 + 10000\underline{a} + 100\underline{b}$
$= 6 + 3a - 4b +$ some multiple of 13.
$\therefore \quad N$ will be divisible by 13 if $(3a - 4b + 6)$ is exactly divisible by 13. (Remember $0 \leq a \leq 9, 0 \leq b \leq 9$).
If $a = 3$, b can be 7;
If $a = 2$, b can be 3;
If $a = 0$, b can be 8;
If $a = 3$, b cannot be a whole number.

Answer is (C).

10. If the numbers 2^{2001} and 5^{2001} are written one after another (in decimal notation) then the total number of digits written altogether is

(A) 2001 (B) 2002 (C) 7^{2001} (D) 10^{2001}

Solution: Let 2^{2001} have p digits when written in full. 5^{2001} have q digits when written in full.

$\therefore \quad 10^{p-1} < 2^{2001} < 10^{p}$

$\therefore \quad 10^{q-1} < 5^{2001} < 10^{q}$

$\therefore \quad 10^{p+q-2} < 10^{2001} < 10^{p+q}$

$\Rightarrow \quad p+q-2 < 2001 < p+q$

$\therefore \quad 2001 = p+q-1$

or $\quad p+q = 2002.$

Answer is (B).

11. Consider the series $1-4+7-10+3-\cdots$ where the $(2n)^{th}$ term is $-2(3n-1)$ and the $(2n+1)^{th}$ term is $(6n+1)$. The sum to first 2001 terms is

(A) 2001 (B) 3001 (C) 304001 (D) -301001

Solution: Taking $n = 1000$, the sum is

$$\underbrace{(1+7+13+19+\cdots+6001)} - \underbrace{(4+10+16+\cdots+5998)}$$

$$= \frac{1001}{2}(6001+1) - \frac{1000}{2}(5998+4) = 3001.$$

The Answer is (B).

12. There are 20 people around a table. Each of them shakes hands with the people to his or her immediate right and immediate left. The number of handshakes that take place is

(A) 20 (B) 40 (C) 41 (D) 111

Solution: There are 20 gaps and each gap corresponds to

a handshake. Thus on the whole the number of handshakes is 20. The Answer is (A).

13. All odd numbers from 1 to 99 (both inclusive) are multiplied together the two right most digits of the product is

(A) 05 (B) 25 (C) 45 (D) 65

Solution: The last two digits will have to be either 25 or 75.

Thus answer is (B).

14. f is a real function such that $f(x+y) = f(xy)$ and $f(\frac{-2}{3}) = \frac{2}{3}$ for all x, y. Then $f(2001)$ is

(A) 2001 (B) $-\frac{2001}{3}$ (C) $\frac{2001}{3}$ (D) $\frac{2}{3}$

Solution: Suppose $f(0) = k$, say.
$x = 0$ gives $f(0+y) = f(0)$ which $\Rightarrow$ $f(y) = k$.
$y = 0$ gives $f(x+0) = f(0)$ which $\Rightarrow$ $f(x) = k$.
Thus $f(x) = f(y)$, for all xy.
$\therefore$ f is a constant function.
Since $f(-\frac{2}{3}) = \frac{2}{3}$, $f(2001)$ is also $\frac{2}{3}$.
Answer is (D).

15. Between 5 p.m. and 6 p.m., I looked at my watch. Mistaking the hour hand for the minute hand, I took the time to be 57 minutes earlier than the correct time. The correct time is actually was

(A) 5-03 (B) 5-58 (C) 5-24 (D) 5-22

Solution: Minute hand should be between 4 and 5. Hour hand should be between 5 and 6. Let the hour hand be x minute - divisions beyond 5. Therefore actual time $= 12x$ minutes past 5. But it was taken to be $(x + 25)$ minutes past 4. Difference $= (60 + 12x) - (x + 25) = 11x + 35$. Thus $11x + 35 = 57$, so $x = 2$. Correct time is 24 minutes past 5.

$\therefore$ Answer is (C).

16. In a school whose student strength is 500, two-thirds of the students who do not wear spectacles do not bring lunch. Three quarters of the students who do not bring lunch do not wear spectacles. On the whole, 60 students wear spectacles and also bring lunch. The number of students who do not wear spectacles and who do not bring lunch is

(A) 360 (B) 320 (C) 240 (D) 280

Solution: Let $X =$ Set of students who do not wear spectacles, $Y =$ Set of students who do not bring lunch and $Z = (X \cap Y)$
Take $n(X) = x, n(Y) = y$ and $n(Z) = z$. Then

$$\frac{2x}{3} = z \tag{1}$$

$$\frac{3y}{4} = z \tag{2}$$

$$500 + z - x - y = 60 \tag{3}$$

(Since $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$)
(1) and (2) $\Rightarrow 8x = 9y = 12z$. This when used in (3) gives $x = 360, y = 320$ and $z = 240$. Answer is (C).

17. The number of real roots of the equation
$2X^{2001} + 3X^{2000} + 2X^{1999} + 3X^{1998} + \cdots + 2X + 3 = 0$ is
(A) 1 (B) 2 (C) 2001 (D) Infinite
Solution: The given equation is

$$(2x + 3)(x^{2000} + x^{1998} + \cdots + x^2 + 1) = 0.$$

One root is $\frac{-3}{2}$ which is real. The other roots are given by

$$x^{2000} + x^{1998} + \cdots + x^2 + 1 = 0.$$

But all the roots of this are imaginary, since all the terms are positive and for no real x their sum can be zero.

$\therefore$ Answer is (A).

18. In solving a quadratic equation, one student copied the equation wrongly making an error only in the constant term and obtained 8,2 as roots; another student also copied the equation wrongly making an error only in the coefficient of the first degree term and obtained $-9, -1$ as roots. The correct equation was
(A) $X^2 - 10X + 9 = 0$ (B) $X^2 + 10X + 9 = 0$
(C) $X^2 - 10X + 16 = 0$ (D) $X^2 + 10X - 16 = 0$

Solution: Equation with roots 8,2 is
$(x - 8)(x - 2) = 0$ i.e. $x^2 - 10x + 16 = 0$ (1)
Equation with roots $-9, -1$ is $(x + 9)(x + 1) = 0$
i.e. $x^2 + 10x + 9 = 0$ (2)

The constant term of (1) is wrong; x term of (2) is wrong.
$\therefore$ The correct equation is $x^2 - 10x + 9 = 0$.
Answer is (A).

19. If $x \leq y \leq z$ are positive integers such that $\left(\frac{1}{x}\right) + \left(\frac{1}{y}\right) + \left(\frac{1}{z}\right) = \frac{7}{15}$ then x cannot be

(A) 7 (B) 3 (C) 4 (D) 5

Solution:

$$\frac{1}{x} \geq \frac{1}{y} \geq \frac{1}{z}$$

$$\therefore \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{3}{x}$$

$$\text{i.e.,} \quad \frac{7}{15} \leq \frac{3}{x}$$

$$\text{or} \quad x \leq \frac{45}{7}$$

$$\therefore \quad x < 7$$

Answer is (A).

20. Men and women devotees totalling 12, men outnumbering women, visited a Swamiji and sought his blessings. The Swamiji distributed 142 flowers. Each man got same number of flowers, while each woman got two more than each man. How many flowers did each woman get?

(A) 13 (B) 11 (C) 12 (D) 9

Solution: Let $M =$ Number of men.

$x =$ Number of flowers each man got.

$Mx + (12 - M)(x + 2) = 142$. $\therefore \quad 6x - M = 59$

or $$x = 9 + \frac{5+M}{6} \qquad (1)$$

M, x are whole numbers; also $M >$ No. of women. i.e., $6 < M < 12$. $\therefore \quad M = 7$ by (1) and $x = 11$.

$\therefore$ Each woman got $x + 2 = 11 + 2 = 13$ flowers.

Answer is (A).

21. For a class, copies of 9 Mathematics books and 16 Science books cost Rs. 220/-. Each book costs an integral number of rupees. Then the cost of each Mathematics book is

(A) Rs.6 (B) Rs.5 (C) Rs.12 (D) Rs.7

Solution: Let Rs. m = cost of one Mathematics book and Rs. s = cost of one science book. s = cost of one Science book.

$\therefore \quad 9m + 16s = 220.$ or $m = 25 - \frac{5+16s}{9}$

As m is a natural number less than 25, $5+16s$ is a multiple of 9. This happens for $s = 7$. $\therefore \quad s = 7$ which $\Rightarrow \quad m = 25 - 13 = 12.$

Answer is (C).

22. The real line $y = mx + c$ is such that $mc > 0$. Then the point which cannot lie on this line is

(A) $(0, 2001)$ (B) $(0, -2001)$ (C) $(19, 99)$ (D) $(2001, 0)$

Solution: $mc > 0 \quad \Rightarrow \quad m, c$ are both positive or both negative.

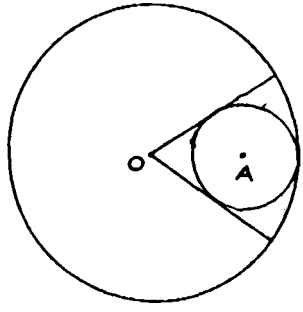
In either case, the line slopes away from the positive side of the x axis.

$\therefore \quad$ The answer is (2001,0). Answer is (D).

[Note: (0,2001) lies on $y = x + 2001$; $(0, -2001)$ is on $y = -x - 2001$; (19,99) lies on $y = 5x + 4$.]

23. Circle A is inscribed in a $60°$ sector of circle with centre O as shown. Then the ratio of the length of the radius of circle with centre O to the radius of circle with centre A is

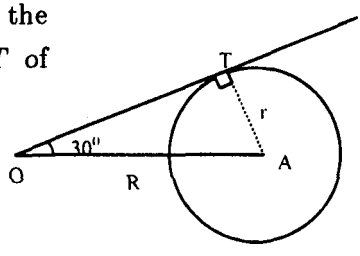
(A) 2:1 (B) $\sqrt{3}:2$ (C) 3:2 (D) 3:1



Solution: Clearly from the $30-60-90$ triangle OAT of the figure, $R = 2r$.

$\therefore \quad \frac{R}{r} = \frac{2}{1}$.

Answer is (A).



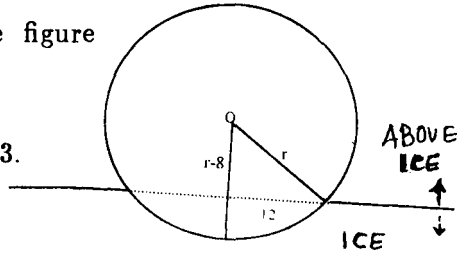
24. A ball was floating in a lake in the Himalayas, when the lake froze. The ball was removed (without breaking the ice), leaving a hole 24 cm across at the top and 8 cm deep. The radius of the ball (in cm) was

(A) 13 (B) $8\sqrt{3}$ (C) $6\sqrt{6}$ (D) 12

Solution: From the figure $(r-8)^2 + 12^2 = r^2$

$\Rightarrow \quad r = \frac{64+144}{16} = 13.$

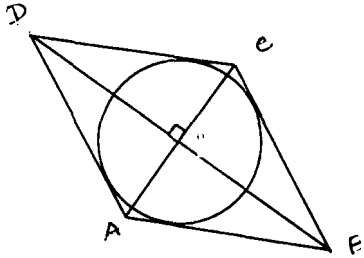
Answer is (A).



25. The radius of a circle inscribed in a rhombus whose diagonals are of length 16 and 30 is

(A) 16 (B) $\frac{480}{17}$ (C) $\frac{240}{17}$ (D) 17

Solution: Area of $\triangle AOB = \frac{1}{2} \times 16 \times 30 = 240$.
Also, $AB = \sqrt{16^2 + 30^2} = 34$

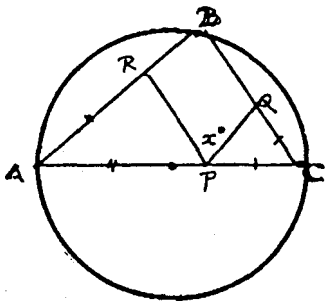


$$\therefore \quad \frac{1}{2}h \times 34 = 240$$

$$h = \frac{240 \times 2}{34} = \frac{240}{17}$$

Answer is (C).

26. In the figure AC is a diameter of the circle. Q and R are points on BC and AC respectively, such that $PC = QC$ and $PA = RA$. Then the measure of the angle QPR is



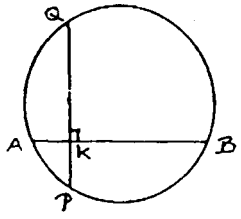
(A) 30° (B) 45° (C) 60° (D) 90°

Solution: Let $m\angle APR = y°$ and $m\angle CPQ = z°$. Then

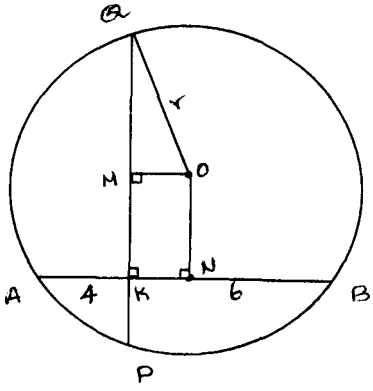
$$\begin{aligned} 180° &= y + z + x \\ &= \frac{180° - A}{2} + \frac{180° - C}{2} + x \\ &= 180° + x - \left(\frac{A + C}{2}\right) \\ &= 180° + x - 45° \end{aligned}$$

$$\therefore \quad x = 45°$$ Answer is (B).

27. In the circle shown, chords AB and PQ meet at K and are perpendicular to one another. If $AK = 4, KB = 6$ and $PK = 2$ then the area of the circle is



(A) 45π (B) 48π (C) 50π (D) 55π



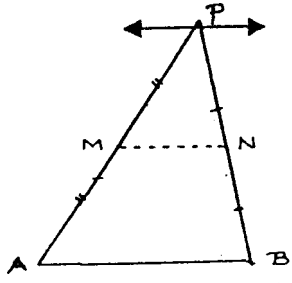
Solution: Since $KA \cdot KB = KP \cdot KQ$, we get $KQ = 12$. $\therefore$ If we drop perpendiculars OM to PQ and ON to AB, then $KMON$ being a rectangle,

$OM = KN = KB - NB = 6 - NB = 6 - 5 = 1 \quad MK = QK - QM = 12 - 7 = 5.$

$\therefore \quad r^2 = QM^2 + MO^2 = 7^2 + 1^2 = 50.$

$\therefore$ Area $\pi r^2 = 50\pi$. Answer is (C).

28. Points M and N are the midpoints of sides PA and PB of triangle PAB. As P moves along a line that is parallel to side AB, consider the following:



I. The length MN
II. The area of triangle PAB
III. The area of trapezoid $ABNM$
IV. The perimeter of triangle PAB

Which of the above will change?

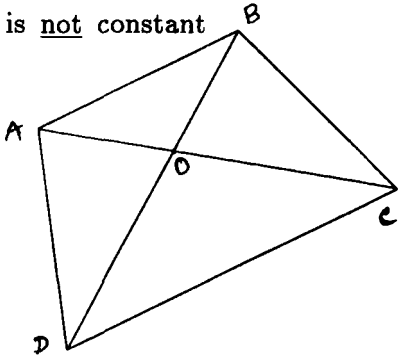
(A) I and II only (B) II and III only
(C) II only (D) IV only

Solution:

I. Length MN will be constant because it is always half of AB in this case.
II. Since P moves parallel to AB, the 'height' of the triangle is constant and so area will be constant.
III. Because $MN||AB$ and $MN = \frac{1}{2}AB$, the area of trapezoid $ABNM$ is also constant.
IV. There is no guarantee that $PA+PB$ will be constant. So perimeter of $\triangle PAB$ is <u>not</u> constant

Therefore, the answer is (D).

29. In the given figure $ABCD$ is a quadrilateral whose diagonals meet at O. If $\triangle AOB, \triangle DOC$ and $\triangle BOC$



have areas 3,10 and 2 respectively, then the area of $\triangle AOD$ is

(A) $\sqrt{30}$ (B) $\sqrt{15}$ (C) $\sqrt{60}$ (D) $\sqrt{20}$

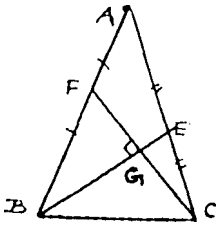
Solution: $\triangle AOD$ and $\triangle COD$ have same attitudes from D. If the area of $\triangle AOD$ is k, (say), then

$$k : 10 = AO : OC$$

$$\begin{aligned} \text{But} \quad AO : OC &= \triangle AOB : \triangle BOC \\ &= 3 : 2 \\ \text{Therefore} \quad k : 10 &= 3 : 2 \\ k &= 15 \end{aligned}$$

$\therefore$ Answer is (B).

30. Medians BE and CF of $\triangle ABC$ are perpendicular. If $BE = 12$ and $CF = 18$, then the area of $\triangle ABC$ is



(A) 36 (B) 72 (C) 93 (D) 144

Solution: Let G be the centroid, the meeting point of the medians. $BG = \frac{2}{3}$ times $BE = \frac{2}{3} \times 12 = 8$.

$\therefore$ Area of $\triangle BCF = \frac{1}{2} \times CF \times BG = \frac{1}{2} \times 18 \times 8 = 72$.

$\therefore$ Area of $\triangle ABC = 2 \times$ area of $\triangle BCF$

$= 2 \times 72 = 144$.

Answer is (D).

SOLUTIONS COMPILED BY: KUMARI R. VIJAYALAKSHMI, T.Nagar, Chennai-17.

AMTI

SCREENING TEST - INTER LEVEL - 2001

Note:

1. *Attempt all questions*
2. *For each question choose the most appropriate answer*
3. *Write the answer in the response sheet supplied separately*
4. *You may use a rough sheet, if necessary, to do calculation*
5. *Calculator/Log tables are not to be used*
6. *In the case of doubts regarding the mode of answering ask your supervisor for clarification*
7. THE TIME ALLOTTED FOR ANSWERING IS 2 HOURS ONLY
8. *Figures in the questions are not drawn to scale*

1. If $|x-2| = p$ where $x < 2$, then $x - p =$

(A) -2 (B) $2 - 2p$ (C) 2 (D) $2p - 2$

Solution: $|x-2| = p$, $x < 2 \Rightarrow 2 - x = p \Rightarrow x = 2 - p \Rightarrow x - p = 2 - 2p$ and hence the solution is (B).

2. Given $ab = a + b$, where a, b are non-zero real numbers, the value of $\frac{a}{b} + \frac{b}{a} - ab$ is

(A) -2 (B) 2 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

Solution:

$$\begin{aligned} \frac{a}{b} + \frac{b}{a} - ab &= \frac{a^2 + b^2}{ab} - (a + b) \\ &= \frac{a^2 + b^2}{a + b} - (a + b) \end{aligned}$$

$$= \frac{(a^2+b^2)-(a+b)^2}{(a+b)}$$

$$= \frac{-2ab}{a+b} = \frac{-2ab}{ab} = -2.$$

and hence the solution is (A).

3. Given $f\left(\frac{x}{4}\right) = 4x^2 - 1$, the values of x for which $f(4x) = 0$ is

(A) 32 or $-\frac{1}{32}$ (B) $-32, -\frac{1}{32}$ (C) $\frac{1}{32}, -32$ (D) $\frac{1}{32}, -\frac{1}{32}$

Solution: $f(\frac{x}{4}) = 4x^2 - 1 = 64(\frac{x}{4})^2 - 1$

$\therefore \quad f(4x) = 0 \Rightarrow 64 \times (4x)^2 - 1 = 0$

$\Rightarrow [(8 \times 4x) - 1]\,[8 \times 4x + 1] = 0$

$\Rightarrow x = \frac{1}{32}$ or $\frac{-1}{32}$.

Solution is D.

4. If for $x, y, z \geq 0, x + y + z = 11$, then the maximum of $xyz + xy + yz + xz$ is

(A) 100 (B) 88 (C) 84 (D) 68

Solution: $x, y, z \geq 0 \quad x + y + z = 11$

$xyz + xy + yz + zx = (x+1)(y+1)(z+1) - (x+y+z) - 1$

$= pqr - 12$ and $p+q+r = 14$ where $p = x+1, q = y+1, r = z+1$.

pqr is max subject to $p+q+r = 14$, when $p = q = r = \frac{14}{3}$.

$\therefore$ Reqd. max $= \frac{14\times14\times14}{27} - 12 = \frac{2744-324}{27} = \frac{2420}{27} = 89\frac{17}{27}$

This is not one of the alternatives. However if x, y, z are restricted to be integers p, q, r are integers. Their values for max. of $pqr - 12$ when $p + q + r = 14$ should be 5,5 and 4.

Max. value $= 100 - 12 = 88$. Answer is (B).

5. When the Mean, Median and Mode of the list $11, 3, 6, 3, 5, 3, x$ are arranged in ascending order, they form a non-constant arithmetic progression. The number of possible values of x is

(A) 0 (B) 1 (C) 2 (D) 3

Solution: Arranging the list of numbers other than x in ascending order, we get

$$3, 3, 3, 5, 6, 11$$

If $x \leq 3$, then both mode and median measure will be 3, which contradicts the statement of the problem.

Case I. If $3 < x \leq 5$, then mode is 3, median is x and mean is $\frac{31+x}{7}$

(i) Thus if mode, median and mean in that order form an AP, then $x - 3 = \frac{31+x}{7} - x$ (1)

(ii) or if mode, mean and median in that order form an AP, then $\frac{31+x}{7} - 3 = x - \frac{31+x}{7}$ (2)

By (1) we have

$$\begin{aligned} 7x - 21 &= 31 + x - 7x \\ \Rightarrow 13x &= 52 \text{ or } x = 4 \end{aligned}$$

or by (2) we have

$$\begin{aligned} 31 + x - 21 &= 7x - 31 - x \\ \Rightarrow 5x &= 41 \\ x &= 8\frac{1}{5}, \text{ inadmissible as } x \leq 5. \end{aligned}$$

Case II. If $x > 5$, then mode is 3, median is 5 and mean is $\frac{31 + x}{7}$.

If mode, median and mean form an AP in that order, we have

$$\begin{aligned} 5 - 3 &= \frac{31 + x}{7} - 5 \\ \Rightarrow x &= 18 \end{aligned} \qquad (1)$$

If mode, mean and median form an AP in that order, we have

$$\begin{aligned} \frac{31 + x}{7} - 3 &= 5 - \frac{31 + x}{7} \\ \Rightarrow x &= -3, \text{ inadmissible as } x > 5. \end{aligned}$$

Thus x can take only two values, namely 4 and 18. Thus the solution is (C).

6. A function f is defined as follows for all integers:

$$f(n) = \begin{cases} n + 5 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$$

Then the number of values of k for which $f(f(f(k))) = 10$ is

(A) 2 (B) 3 (C) 4 (D) 5

Solution:

$$\begin{aligned} & f(f(f(k))) &= & \ 10 \\ \Rightarrow \quad & f(f(k)) &= & \ 5 \text{ or } 20 \\ \Rightarrow \quad & f(k) &= & \ 10 \text{ or } 15 \text{ or } 40 \\ \Rightarrow \quad & k &= & \ 5 \text{ or } 20 \text{ or } 30 \text{ or } 35 \text{ or } 80. \end{aligned}$$

Thus k can take 5 values. Solution is (D).

7. Number of positive integers b for which $\log_b 1024$ is also a positive integer is

(A) 4 (B) 3 (C) 2 (D) 1

Solution: $1024 = 2^{10}$ or 4^5 or 32^2 or 1024^1.

Thus $\log_b 1024 = 1$ when $b = 1024$

$\log_b 1024 = \log_b 32^2 = 2\log_b 32 = 2$ when $b = 32$, $\log_b 1024 = \log_b 4^5 = 5\log_b 4 = 5$ when $b = 4$ and finally $\log_b 1024 = \log_b 2^{10} = 10\log_b 2 = 10$ if $b = 2$ and thus for $b = 1024, 32, 4$ or 2, the value of $\log_b 1024$ is also an integer and the solution is (A).

8. A teacher gave a test to six students. He entered their marks in a random order in a computer, which recalculated the average after each mark was entered. The average thus found was always an integer. The marks scored in ascending order were 41,46,50,52,56 and 61. The last of these marks entered by the teacher was

(A) 56 (B) 50 (C) 46 (D) 61

Solution: The sum of the first two marks entered should be an even number, first four a multiple of 4 and first five a multiple of 5. So, if the first two marks entered are 41 and 61, since none other is a multiple of 3, the sum of the first 3 cannot be a multiple of 3. For the same reason, the first two numbers entered cannot be 46,50 or 50 and 52, or 46 and 52, or 50 and 56. So, 46,52,61 can satisfy the condition for triplets of numbers.

If we consider the triplets 45,52,61, the fourth number entered should make the sum of these 4 numbers divisible by 4. and clearly it should be an odd number; thus 46,52,61,41 (the sum being 200)satisfy the condition; so also 50,56,41 and 61 (the sum being 208).

But for the first list of 4 numbers 46,52,61,41 we should add a multiple of 5, so that the five numbers sum up to a multiple of 5. But there is none in the list. So we should consider the second list of 50,56,41 and 61 and the fifth number should be 52 (just there are two numbers left out 52 and 46).

So, the last number should be 46 and the solution is (C).

9. In the year N, the 200^{th} day was a Thursday; and in the next year (that is in the year $N+1$), the 100^{th} day was also a Thursday. Then the 300^{th} day in the year $N-1$ should have been

(A) Thursday (B) Friday (C) Saturday (D) Sunday

Solution: In any calendar you might have seen that every 8th day is the same (as that of first day).

In the year N, 200^{th} day is $(200 = 7 \times 28 + 4)$ a Thursday.

In the year $N+1$, 100^{th} day is $(365 + 100)^{th}$ day from from January first of year N, if N is not a leap year or $(366 + 100)^{th}$ day if N is a leap year.

$465 = 7 \times 67 + 3$ cannot be a Thursday as any rank of the day is of the form $7k + 4$, only then it will be a Thursday.

$466 = 7 \times 67 + 4$ can be a Thursday as given.

Thus year N is a leap year.

In a leap year, January 1^{st} and December 31^{st} will be two consecutive days.

Since $366 = 7 \times 52 + 2$, 31^{st} of December in the year N is a Tuesday.

1^{st} January is a Monday [as N was found to be a leap year] and 31^{st} December of year $N - 1$ is a Sunday.

As $365 = 7 \times 52 + 1$ and 365^{th} day is a Sunday,

300^{th} day should therefore be a Friday and the solution is (B) $(300 = 7 \times 42 + 6)$.

10. If $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$ and $z + \frac{1}{x} = 1\frac{2}{3}$, $x, y, z \neq 0$,

the value of xyz is

(A) 1 (B) -1 (C) $\pm i$ (D) ± 1

Solution:

$$\left(x+\frac{1}{y}\right), \left(y+\frac{1}{z}\right)+\left(z+\frac{1}{x}\right)=4+1+\frac{5}{3}=\frac{20}{3}.$$

$\frac{20}{3}$ is also equal to

$$\left(x+\frac{1}{y}\right)\left(y+\frac{1}{z}\right)\left(z+\frac{1}{x}\right)$$

$$= xyz+x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{xyz}$$

$$= xyz+\frac{1}{xyz}+\frac{20}{3}.$$

So, $xyz+\frac{1}{xyz}=0$

or $x^2y^2z^2+1=0$

or $xyz=\pm i$.

Therefore, solution is (C).

11. Let $E(n)$ denote the number of even digits in n; for example, $E(2)=1, E(19)=0, E(5672)=2$. Then $E[E(101)\times E(201)\times E(301)\times\cdots\times E(2001)]$ is

(A) 1 (B) 2 (C) 3 (D) 4

Solution: In the sequence

$$101, 201, 301, \cdots, 901, 1001, 1101\cdots 1901, 2001,$$

there are $\frac{2001 - 101}{100} + 1 = \frac{1900}{100} + 1 = 20$ terms

The number of even numbers in these terms are alternatively 1 and 2 upto 1901 and for 2001 it is 3. Thus

$E(101) \times E(201) \times \cdots \times E(2001) = 1 \times 2 \times 1 \times 2 \times \cdots 1 \times 3.$
$= 2^9 \times 1^{10} \times 3 = 512 \times 3 = 1536$
$\therefore\ E[E(101) \times 6(201) \times \cdots 6(2001)]$
$= E[1536] = 1$
and the solution is (A).

12. If $x^2 + y^2 - 14x - 6y - 6 = 0$, the largest possible value of $3x + 4y$ is

(A) 72 (B) 73 (C) 74 (D) 75

Solution:

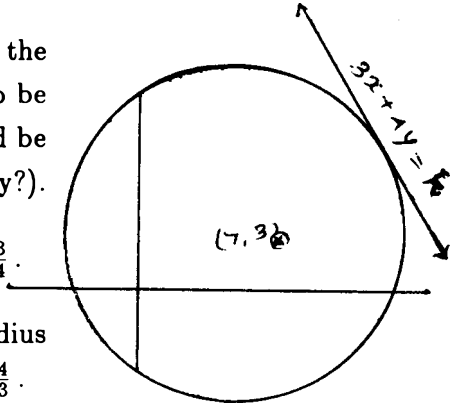
$$x^2 + y^2 - 14x - 6y - 6y - 6 = 0$$
$$\Rightarrow\ (x - 7)^2 + (y - 3)^2 = 8^2$$

and clearly, the equation represents a circle centre (7,3) and radius 8.

Consider the equation of the line $3x + 4y = k$ for k to be a maximum the line should be a tangent to the circle (why?).

The slope of the line is $-\frac{3}{4}$.

Hence the slope of the radius at the point of contact is $\frac{4}{3}$.



Thus the equation of the radius is

$$\begin{aligned} y - 3 &= \frac{4}{3}(x - 7) \\ \Rightarrow \quad y &= \frac{4x - 19}{3} . \end{aligned}$$

The maximum value of $3x + 4y = k$ is corresponding to one of the two values of (x, y) which are the points of contact of the tangents.

The x co-ordinate of the points of contact are given by

$$\begin{aligned} (x-7)^2 + \left(\frac{4x-19}{3} - 3\right)^2 &= 8^2 \\ \Rightarrow \; 3^2(x-7)^2 + 4^2(x-7)^2 &= 3^2 \times 8^2 \\ \Rightarrow \; 5^2(x-7)^2 &= 3^2 \times 8^2 \\ \text{or} \quad x - 7 &= \pm\sqrt{\frac{3^2 \times 8^2}{5^2}} = \pm\frac{24}{5} \end{aligned}$$

$\therefore \quad x = 7 + \frac{24}{5}$ or $7 - \frac{24}{5}$. i.e., $\frac{59}{5}$ or $\frac{11}{5}$

The corresponding y values are $\frac{47}{5}$ and $\frac{-17}{5}$.

The the points of contact of the tangents are $\left(\frac{59}{5}, \frac{47}{5}\right)$ and $\left(\frac{11}{5}, \frac{-17}{5}\right)$.

Clearly the values $x = \frac{59}{5}, y = \frac{47}{5}$ give the maximum value for $3x + 4y$ which is

$$\frac{3 \times 59}{5} + \frac{4 \times 47}{5} = \frac{365}{5} = 73$$

the value being -7 when $x = \frac{11}{5}, y = \frac{-17}{5}$.

The answer is (B).

Another Solution:

Consider $x^2+y^2-14x-6y-6=0$ (1) and $3x+4y=k$. (2)

Substituting $y=\frac{k-3x}{4}$ in (1),

we get $4^2x^2+(k-3x)^2-4^2\times 14x-4\times 6(k-3x)-6\times 4^2=0$
$\Rightarrow \quad 25x^2-2(3k+76)x+(k^2-24k+96)=0.$
For real roots the discriminant

$$4(3k+76)^2-4\times 25(k^2-24k-96)\geq 0$$

i.e., $-(k-73(k+7)\geq 0$ or $(k-73(k+7)\leq 0.$
$\Rightarrow \quad -7\leq k\leq 73.$ The maximum value of k is 73.

13. A lady takes her triplets and a younger niece to a restaurant on the birthday of the triplets. The restaurant charges Rs.75 for the mother and Rs.5 for each completed year of a child's age. If the total bill is Rs.160 and if her niece is younger than her triplets, the age of the triplets is

(A) 14 (B) 11 (C) 8 (D) 5

Solution: The bill for triplets and niece is Rs. (160 − 75) = Rs. 85.

If the age in completed years of triplets is x and that of her niece is y
$5(3x+y)=85$ and $x>y \Rightarrow 3x+y=17.$
Clearly, $x\leq 5$. If $x=5$, $3\times 5+2=17$ and $5>y=2$; but for $x=4$, we get $3\times 4+5=17$ and $3<y=5$. Thus for $n\leq 4$, the age of the niece is not less than that of

the triplet. So the age of the triplets is 5 and the solution is (D).

14. The integer $4n$ has 40 divisors and the integer $9n$ has 48 divisors. The number of divisors of $6n$ is

(A) 36 (B) 39 (C) 42 (D) 45

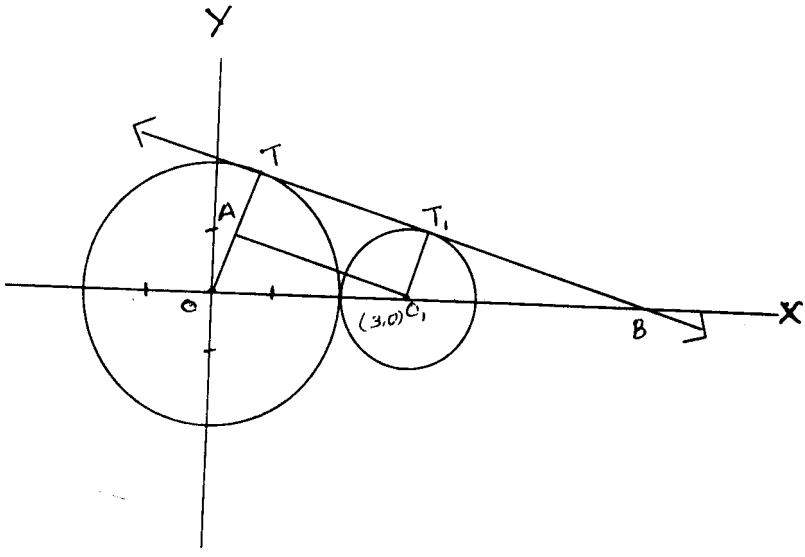
Solution: If $n = 2^{\alpha-1}3^{\beta-1}N$ where N is not divisible by 2 or 3 and if $d(N) = x$, then $d(n) = \alpha\beta x$.

$\therefore \quad d(4n) = (\alpha+2)\beta x = 40, d(9n) = \alpha(\beta+2)x = 48$ and
$d(6n) = (\alpha+1)(\beta+1)x = (\alpha\beta+\alpha+\beta+1)x$
$d(4n) + d(9n) = 2(\alpha\beta+\alpha+\beta)x = 88$
$\therefore \quad (\alpha\beta+\alpha+\beta)x = 44.$
$\therefore \quad d(6n) = 44 + x$
$d(9n) - d(4n) = 8 = 2(\alpha-\beta)x$
$\therefore \quad (\alpha-\beta)x = 4.$
$\therefore \quad \alpha-\beta = 4, x = 1$ or $\alpha-\beta = 1, x = 4.$
$\therefore \quad d(6n) =$ 45 or 48.

Note: The other possible solution is $d(6n) = 48$, but it is not included in the multiple choice.

15. Two circles, centre (0,0)with radius 2 units and centre (3,0) with radius 1 unit, have a common tangent drawn passing through the first quadrant. The slope of the tangent is:

(A) $-\frac{\sqrt{2}}{4}$ (B) $\frac{\sqrt{2}}{4}$ (C) $\frac{3}{\sqrt{2}}$ (D) $\frac{-3}{\sqrt{2}}$



Solution: From the figure, you find O_1ATT_1 is a rectangle and $OO_1 = 3$ units, $OA = OT - AT = OT - OT_1 = 2 - 1 = 1$ and hence $O_1A = \sqrt{3^2 - 1^2} = \sqrt{8}$.

$\tan \angle OO_1A = \frac{1}{\sqrt{8}}$ or $\frac{\sqrt{2}}{4}$ and hence the slope of $AO_1 = \tan(AO_1B) = \tan(180 - OO_1A) = -\tan OO_1A = -\frac{\sqrt{2}}{4}$ and since the common tangent TT_1 is parallel to AO_1, slope of the common tangent $= \frac{-\sqrt{2}}{4}$ and the solution is (A).

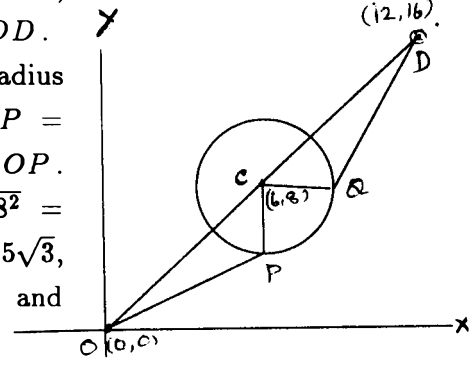
16. The length of the shortest path from the origin to (12,16) that does not go inside the circle $x^2 + y^2 - 12x - 16y + 75 = 0$ is

(A) $10\sqrt{3}$ (B) $20\sqrt{3}$ (C) $10\sqrt{3} + \frac{5\pi}{3}$ (D) $20\sqrt{3} + \frac{5\pi}{3}$

Solution: The given equation represents a circle and can be re-written as $(x - 6)^2 + (y - 8)^2 = 25$ and hence the centre

C is $(6,8)$, and the radius is 5 units. Since the path should not go inside the circle, the path consists of two tangents drawn from origin O and $(12,16)$ to the circle and the arc from P to Q where P and Q are the points of contact of the two tangents.

Clearly O,C,D are collinear, with C as midpoint of OD. $OC = 10$ units, the radius $CP = 5$ and hence $OP = 5\sqrt{3}$ units as $CP \perp OP$. Also as $CD = \sqrt{6^2+8^2} = 10, CQ = 5$, and $DQ = 5\sqrt{3}$, $\angle COP = 30° = \angle CDQ$ and hence $\angle PCQ = 60°$.



So the length of arc $PQ = \frac{5}{3}\pi$ and hence the length of the path is $\frac{5}{3}\pi + 10\sqrt{3}$. The solution is (C).

Note. In the question paper instead of $\frac{5\pi}{3}$ it was given as $\frac{5\pi}{\sqrt{3}}$.

17. Three boxes A,B,C each contain 4 white balls and 5 black balls. All the balls are identical except for the colour. A ball is shifted from box A to box B and then a ball is shifted from box B to box C and finally a ball is shifted from box C to box A. The probability that each of the boxes will contain again 4 white balls and 5 black balls is

(A) $\frac{14}{45}$ (B) $\frac{4}{5}$ (C) $\frac{16}{25}$ (D) $\frac{1}{45}$

Solution: To satisfy the condition, if white ball is transferred from box A to box B, then white ball to be transferred from box B to box C and again white ball is to be transferred back to box A from box C.

(or) All balls of the shifts should be black. Thus the probability that each box has the same position after the shifts is

$$\underset{\substack{\text{[if white ball is}\\ \text{shifted each time]}}}{\frac{4}{9}\times\frac{5}{10}\times\frac{5}{10}} + \underset{\substack{\text{[if balck ball is}\\ \text{shifted each time]}}}{\frac{5}{9}\times\frac{6}{10}\times\frac{6}{10}}$$

$$\text{i.e.}\quad \frac{1}{9}+\frac{1}{5} = \frac{14}{45}$$

and the solution is (A).

18. A fair die is tossed twice and the outcomes in which "the difference between two tosses is 1" are recorded. The probability that the sum of the two tosses is a prime number, out of the recorded outcomes, is

(A) $\frac{4}{5}$ (B) $\frac{1}{9}$ (C) $\frac{2}{5}$ (D) $\frac{4}{9}$

Solution: The sample space of the given random experiment is

$$(6,5),(5,4),(4,3),(3,2),(2,1),(1,2),(2,3),(3,4),(4,5),(5,6)$$

The number of favourable outcomes of the event that the sum is a prime number is 8, and the number of total outcomes is 10 and hence the probability is $\frac{4}{5}$ and the solution is (A).

19. The rooms allotted to the participants of a contest were numbered consecutively beginning with the number 1. The plastic digits costing Rs.1.50 per piece were used for the purpose. If the total cost of the plastic digits was Rs.297, the number of rooms allotted was

(A) 102 (B) 108 (C) 109 (D) 198

Solution: The total number of plastic digits used is $\frac{297}{1\frac{1}{2}} = 198$.

The total number of rooms with single digit numbers is 9.

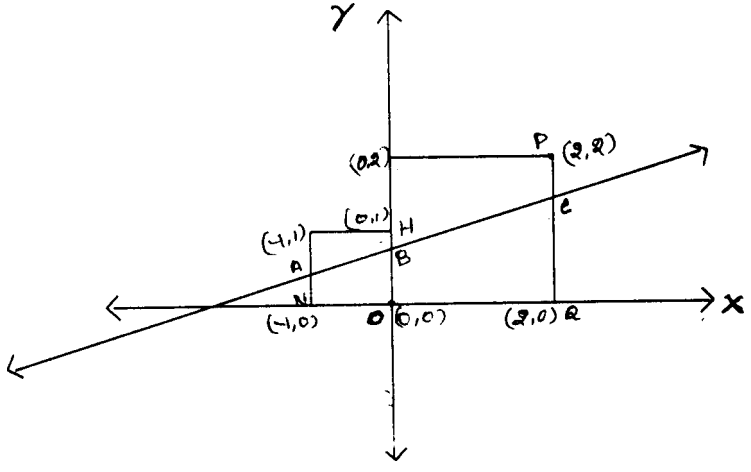
The number of rooms with two digit numbers (10 to 99) 90 and the number of digits used for rooms with two digit numbers= $90 \times 2 = 180$..
Thus upto 99 room the number of plastic digits used = 189.
The number of digits yet to be used is $198 - 189 = 9$.
These 9 digits are used for 3 more rooms, with 3 digit numbers.
Thus total number of rooms is $99+3 = 102$ and the solution is (A).

20. If two squares with vertices $(0,0)$, $(0,1)$, $(-1,1)$, $(-1,0)$ and $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$ are simultaneously bisected by a line, then the equation of the line is

(A) $x + y + 2 = 0$ (B) $3x - y - 2 = 0$
(C) $x - 3y + 2 = 0$ (D) $x + 3y - 2 = 0$

Solution: If l is the line that simultaneously bisects the

squares and if A, B and C are the points of intersection of line and the sides of the squares (see the figure), then one can prove that $NA = MB$ and $OB = PC$.



Let $NA = a$, then $MB = a \Rightarrow OB = 1 - a$ and $PC = 1 - a \Rightarrow QC = (1 + a)$.

Thus the co-ordinates of A, B and C are $(-1, a)$, $(0, 1 - a)$, $(2, 1 + a)$. Since A, B, C are collinear we have

$$\begin{aligned} \frac{(1-a)-a}{0+1} &= \frac{1+a-1+a}{2-0} \\ \text{or} \quad 1-2a &= a \\ \text{or} \quad a &= \frac{1}{3} \end{aligned}$$

The co-ordinates of A, B, C are $(-1, \frac{1}{3}), (0, \frac{2}{3}), (2, 1\frac{1}{3})$. The equation l is therefore $\frac{y-\frac{1}{3}}{\frac{1}{3}} = \frac{x+1}{1}$ or $3y - 1 = x + 1$.

Solution is (C).

21. With the usual notation for a triangle ABC, if $(a+b+c)(a+b-c) = 3ab$, then the angle opposite to side $c = AB$ is

(A) 30° (B) 45° (C) 60° (D) 90°

Solution:

$$\begin{aligned}(a+b+c)(a+b-c) &= 3ab\\ \Rightarrow (a+b)^2 - c^2 - 3ab &= 0\\ \Rightarrow a^2 + b^2 - c^2 - ab &= 0\\ \Rightarrow c^2 = a^2 + b^2 - ab\end{aligned}$$

Applying cosine formula for triangles we get

$$\begin{aligned}a^2 + b^2 - ab &= a^2 + b^2 - 2ab\cos C\\ \Rightarrow \cos C &= \frac{1}{2}\\ \Rightarrow C &= 60°\end{aligned}$$

and C is the angle opposite to AB and hence the solution is (C).

22. Through a point on the hypotenuse of a right angled triangle, lines are drawn parallel to the other two sides, so that the triangle is divided into a square and two triangles. If the area of one of the two small right triangles is m times the area of the square, the ratio of the area of the other triangle to that of the first one is

(A) $\frac{1}{4m^2}$ (B) $4m^2$ (C) $\frac{1}{4m}$ (D) $4m$

Solution: $\triangle$'s APG and QRC are similar

$$\therefore \frac{AP}{PQ} = \frac{QR}{RC} \text{ i.e.} \frac{c-x}{x} = \frac{x}{a-x}$$

$$\text{or } ac-(a+c)x = 0$$

$$\text{or } x = \frac{ac}{a+c}$$

$$\therefore\ c-x=\frac{c(a+c)-ac}{a+c}=\frac{c^2}{a+c}$$

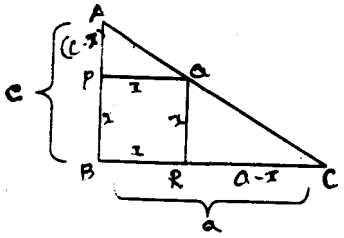
and $a-x=\dfrac{a^2}{a+c}$. Now,

$$\frac{\triangle APQ}{sq\ PQRB} = m$$

$$\Rightarrow \frac{\frac{1}{2}\times\frac{c^2}{(a+c)}\times\frac{ac}{(a+c)}}{\frac{(ac)^2}{(a+c)^2}} = m$$

$$\Rightarrow \frac{c^3a}{a^2c^2} = 2m$$

$$\Rightarrow 2m = \frac{c}{a} \text{ or } c=2ma$$



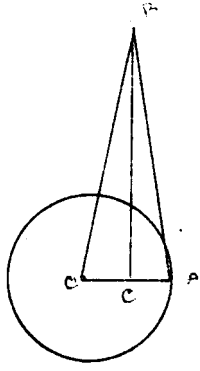
Now
$$\frac{\triangle QRC}{\triangle APQ} = \frac{\frac{1}{2}x\cdot(a-x)}{\frac{1}{2}x\cdot(c-x)}$$

$$= \frac{a-x}{c-x}=\frac{a^2}{c^2}$$

$$= \frac{a^2}{4m^2a^2}=\frac{1}{4m^2}$$

The solution is (A).

23. In the figure, the radius of the circle centred at O is 1 unit and $\angle BOA = \theta$. AB is a tangent to the circle at A. If BC bisects $\angle OBA$, meeting the radius OA at C, then the length of OC is



(A) $\sec^2\theta + \tan^2\theta$ (B) $\frac{1}{(1+\sin\theta)}$

(C) $\frac{1}{(1+\cos\theta)}$ (D) $\operatorname{cosec}^2\theta + \cot^2\theta$

Solution:

$$\begin{aligned} OC &= 1 - CA \\ \angle OBA &= (90 - \theta), \\ \angle CBO = \angle CBA &= \frac{90 - \theta}{2} \\ AB &= \tan\theta \\ CA &= AB\tan\left(\frac{90 - \theta}{2}\right) \\ &= \tan\theta\tan\left(\frac{90^\circ - \theta}{2}\right). \end{aligned}$$

As $$\tan\frac{\alpha}{2} = \frac{1 - \cos\alpha}{\sin\alpha}$$

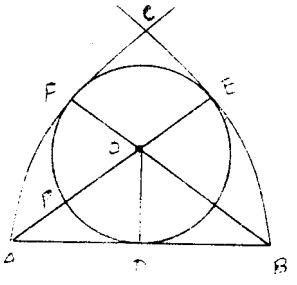
$$\begin{aligned} CA &= \tan\theta \times \frac{1 - \cos(90 - \theta)}{\sin(90 - \theta)} \\ &= \frac{\sin\theta}{\cos\theta} \times \frac{1 - \sin\theta}{\cos\theta} \\ &= \frac{\sin\theta - \sin^2\theta}{\cos^2\theta} \\ OC &= 1 - CA = 1 - \frac{\sin\theta - \sin^2\theta}{\cos^2\theta} \end{aligned}$$

$$= \frac{1-\sin\theta}{\cos^2\theta}$$

$$= \frac{1-\sin\theta}{1-\sin^2\theta} = \frac{1}{1+\sin\theta}$$

The solution is (B).

24. In the figure, the arcs with centres A, B and radii AB intersect at C, as shown. The circle with centre O, touches the segment AB, are BC and arc AC at D, E, F respectively. If the length of arc BC is a cm, then the circumference of the circle centred at O (in cm) is



(A) $\frac{9a^2}{4}$ (B) $\frac{3a}{2}$ (C) $\frac{9a}{4}$ (D) $\frac{3a^2}{2}$

Solution: $AE = BF = AB$ and $OE = OF$ so that $AO = AE - OE = BF - OF = OB$ and hence D is the midpoint of AB.

$$\begin{aligned} AD^2 &= AP \cdot AE \\ &= (AE - PE)AE \\ &= (AE - 2r)AE, \end{aligned}$$

where $PE = 2r$ is the diameter of the smaller circle.

$$\Rightarrow \frac{AB^2}{4} = AE(AE - 2r)$$

$$\Rightarrow \frac{AB^2}{4} = AB(AB - 2r)$$

$$\Rightarrow r = \frac{3}{8}AB.$$

Arc $BC = \frac{1}{6}$ the circumference of the circle with centre A and radius AB [as $AB = BC = AC$]

Circumference of the circle with A as centre and radius $AB = 6 \times$ arc $BC = 6a$.

$$\frac{\text{circumference of the smaller circle}}{\text{circumference of the circle with centre A}} = \frac{\text{radius of the smaller circle}}{\text{radius of the circle with centre A}}$$

$$= \frac{\frac{3}{8}AB}{AB} = 3/8$$

$\Rightarrow$ Circumference of the smaller circle $= \frac{3}{8} \times 6a = \frac{9}{4}a$ cm.
The answer is (C).

25. The number of line segments that have both their end points at vertices of a cuboid is
(A) 12 (B) 24 (C) 28 (D) 36

Solution: Any cuboid has 8 vertices. Two vertices determine a line segment. Thus the number of line segments, with their end points as vertices of the cuboid is ${}_8C_2 = \frac{8 \times 7}{1 \times 2} = 28$ and the solution is (C).

26. A regular polygon of side 2 units is circumscribed about a circle of radius 2 units. The area enclosed between the circle and the polygon is (in square units)
(A) $\pi\sqrt{2}$ (B) π (C) 2π (D) $\pi\sqrt{5}$

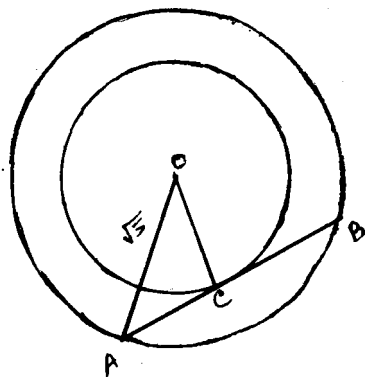
Note: The problem is incorrect. It should be reworded as follows:

Given a circle of radius 2cm, there are infinitely many

line segments of length 2cm, which are tangents to the circle at their midpoints. The area of the region consisting of all such line segments in sq cm is:

(A) $\pi\sqrt{2}$ (B) π (C) 2π (D) $\pi\sqrt{5}$

Solution: Since, AB, one such line segment is tangential to the circle at its mid point C, $OA = \sqrt{5}$ cm and thus every one of these infinitely many line segments has its end points at a distance of $\sqrt{5}$ cm, so these line segments are enclosed in the region of the circular ring (annulus) between the circle of radius 2 cm and radius $\sqrt{5}$ cm.



So the area of the region consisting of these line segments is

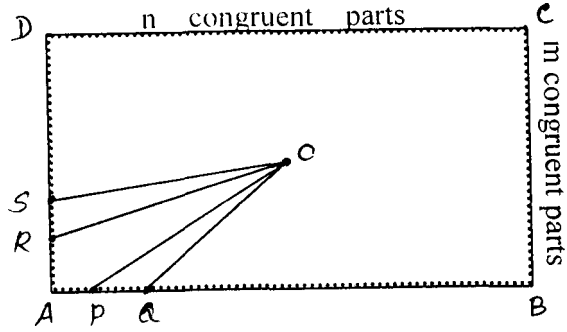
$$\begin{aligned}\left(\pi(\sqrt{5})^2 - \pi \times 2^2\right) \text{cm}^2 &= (5\pi - 4\pi)\ \text{cm}^2 \\ &= \pi\ \text{cm}^2\end{aligned}$$

Answer is (B).

27. $ABCD$ is a rectangle; AB and CD are divided into n congruent segments while AD and BC are divided into m congruent segments. The diagonals AC and BD intersect at O. Also, points P, Q are on AB and R, S are on AD. The segments PQ and RS have 2 parts and 3 parts

respectively. The ratio of area of $\triangle OPQ$ to that of $\triangle ORS$ is

(A) $2m : 3n$ (B) $3n : 2m$ (C) $2 : 3$ (D) $3 : 2$



Solution:

Area of $\triangle OPQ = \frac{1}{2} \times \frac{AB}{n} \times 2 \times \frac{1}{2}AD$
[perpendicular from O to $AB = \frac{1}{2}AD$]
Area of $\triangle ORS = \frac{1}{2} \times \frac{AD}{m} \times 3 \times \frac{1}{2}AB$

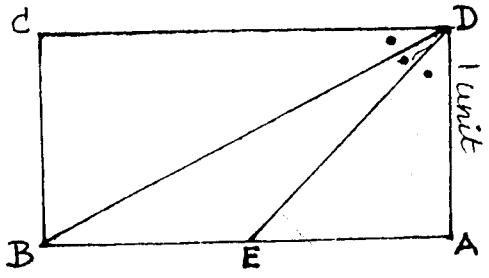
Therefore, the ratio of the areas of $\triangle OPQ$ and $\triangle OSR$ is as

$$\frac{2 \times AB \times AD}{4n} : \frac{3AB \cdot AD}{4m} = 2m : 3n$$

The solution is (A).

28. In the rectangle $ABCD, AD = 1$; E is on AB; and DB, DE trisect $\angle ADC$. Then the difference between the largest side and the sum of the two smaller sides of $\triangle BDE$ is

(A) $\left(\frac{4}{\sqrt{3}}\right) - 2$
(B) $4 - \left(\frac{2}{\sqrt{3}}\right)$
(C) $2 - \left(\frac{2}{\sqrt{3}}\right)$
(D) $4 - \left(\frac{4}{\sqrt{3}}\right)$



Solution: Since $ABCD$ is a rectangle, the angles of $\triangle ADE$ and ADB are 90° , 60° and 30° and 90° , 30° and 60° respectively.

Hence, $EA = \frac{1}{\sqrt{3}}$ unit, $AB = \sqrt{3}$ units and hence $BE = DE = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$ units and $BD = 2$ units.

So, $BE + DE - BD = \frac{2\times 2}{\sqrt{3}} - 2 = \frac{4}{\sqrt{3}} - 2$ units.
Solution is (A).

29. The sequence $1, 2, 1, 1, 2, 2, 1, 1, 1, 2, 2, 2, \cdots$ consists of 1's and 2's in blocks; and the n^{th} block has n ones followed by n twos. The sum of the first 2001 terms of this sequence is

(A) 6004 (B) 6003 (C) 3002 (D) 3001

Solution: Supposing the 2001^{th} term lies in the $(n+1)^{th}$ block, then

$$\begin{aligned} 2(1+2+3+\cdots+n) &< 2001 \\ \frac{2\times n(n+1)}{2} &< 2001 \\ n^2+n-2001 &< 0 \\ \Rightarrow n &\geq 44. \end{aligned}$$

$[44 \times 45 = 1980 < 2001]$

and hence 2001^{th} term lies in the 45^{th} block.

That is upto 21^{th} term in the 45^{th} block which consists of 45 ones followed by 45 twos.

So the sum of the terms required
= sum of all the terms upto 44^{th} block $+21 \times 1$

$= 2 + (2+2) + (2+2+2) + \cdots + (2+2+2+\cdots+2)$
$+ 1 + (1+1) + (1+1+1) + \cdots + (1+1+1+1) + 1 + 1 +$
$\cdots + 1$ (21 ones)
$= 2 + 2 \times 2 + 3 \times 2 + \cdots + 44 \times 2$
$+ (1 + 2 + 3 + \cdots + 44) + 21.$
$= 3[1 + 2 + 3 + \cdots + 44] + 21$
$= \frac{3 \times 44 \times 45}{2} + 21$
$= 2970 + 21 = 2991.$

[Note: This answer is not given in the alternatives.]

30. A sealed cover contains a chit with a single digit number on it. Three of the following statements are true and the others are false:

I The digit is 1
II The digit is 2
III The digit is 3
IV The digit is 4

Then which one of the following answers is incorrect?

(A) I is false (B) IV is true
(C) II is false (D) III is false

Solution: Let us consider the choices. In A, it is given that I is false - It does not contradict, the truth of the statements II, III and IV and hence the conditions of the problem are satisfied. Similarly, the correctness of B and C also satisfy the conditions of the problem. If the choice D is correct, then the "digit is 3" is true and it implies that atleast one of the statements I and II should be true. But this means that the digit is 3 as well as 1 or 2, which is a contradiction. Therefore, the answer in D is incorrect. The solution is (D).

SOLUTIONS COMPILED BY: KUMARI R. VIJAYALAKSHMI, T.Nagar, Chennai-17.

CONTRIBUTIONS TO THE MATHEMATICS TEACHER ARE INVITED. THE PAPERS INTENDED FOR PUBLICATION IN MT MAY KINDLY BE GIVEN IN DUPLICATE, NEATLY TYPED OR LEGIBLY WRITTEN ON ONE SIDE OF THE PAPER ONLY, WITH FIGURES, IF ANY, DRAWN CLEARLY ON SEPARATE SHEETS. ARTICLES ON INNOVATIVE IDEAS ON TEACHING, CLASSROOM EXPERIENCES, NON-ROUTINE PROBLEM SOLVING ETC., ARE WELCOME.

AMTI

JUNIOR LEVEL : FINAL - 2001

Instructions:

1. *Answer as many questions as possible.*
2. *Elegant solutions will receive extra credits.*
3. *Maximum time allotted is 3 hours.*
4. *Attach the Face Slip, duly filled in, and also your rough-work sheets along with answer sheets.*

1. A natural number is good if it can be expressed both as a sum of two consecutive natural numbers and as a sum of three consecutive natural numbers. Show that

(i) 2001 is good but 3001 is not.

(ii) The product of two good numbers is good.

(iii) If the product of two number is good, then at least one of them is good.

Solution:

(i) Since a good number n is the sum of two consecutive integers it must be odd; and also since it is the sum of three consecutive integers it must be divisible by 3. So for some integers m and k,

$$n = 3k = 2m + 1$$

Now, $n = m+(m+1)$ and also $n = (k-1)+k+(k+1)$. Therefore every odd multiple of 3 (bigger than 3) is

"good"; i.e., a number is good iff it is an odd multiple of 3 (> 3). Thus 2001 is good, but 3001 is not!

(ii) The product of any two odd multiples of 3 is again an odd multiple of 3 and hence the product of two good numbers is always good.

(iii) If the product of two natural numbers x and y is good, then xy is an odd multiple of 3. This implies that at least one of them is an odd multiple of 3. Thus if the product of two numbers is good then at least one of them is good.

2. Let A be the set of all 7-digit numbers with different digits $1, 2, 3, 4, 5, 6$ and 7. Prove that there are no two numbers in A such that one of them is divisible by the other.

Solution: Suppose there are two numbers n and m in A such that m divides n. Let $n = mk$. Then we must have

$$2 \leq k \leq \frac{7654321}{1234567} < 7$$

This means that $k \in \{2, 3, 4, 5, 6\}$. Now $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ and therefore no element in A is divisible by 3. This implies that k is not 3 or 6.

Case 1. $k = 2$.

In this case, $n = 2m$ and $n + m = 3m$. But both m and n are 1 modulo 3 since each element of A has sum of its digits equal to 28=1 mod 3. So $m + n = 2 \mod 3$ and therefore $k = 2$ is not possible.

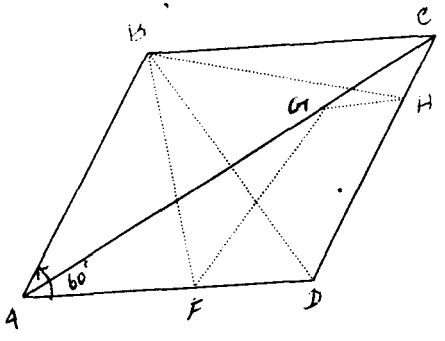
Case 2. $k = 5$.
In this case, $n = 5m$ and $m+n = 6m$. The same reasoning as above says that it is impossible.

Case 3. $k = 4$.
In this case, $n = 4m$ and $n-m = 3m$. Now, $m = n = 28 = 1 \mod 9$ implies that $n - m$ is a multiple of 9. Therefore $n - m = 3m = 9p$ or $m = 3p$ which is impossible.

The above analysis shows that there are no two numbers in A such that one of them is divisible by the other.

3. Given the rhombus $ABCD$ with $\angle A = 60°$. The points F, H and G are marked on the segments AD, DC and the diagonal AC so that $DFGH$ is a parallelogram. Prove that the triangle FBH is equilateral.

Solution: We have $\angle HGC = \angle DAC = \angle ACD = 30°$. Therefore $GH = HC$ and hence $FD = HC$. $\angle A = 60°$ implies that the rhombus $ABCD$ is divided into two equilateral triangles BDA and BDC by the diagonal BD



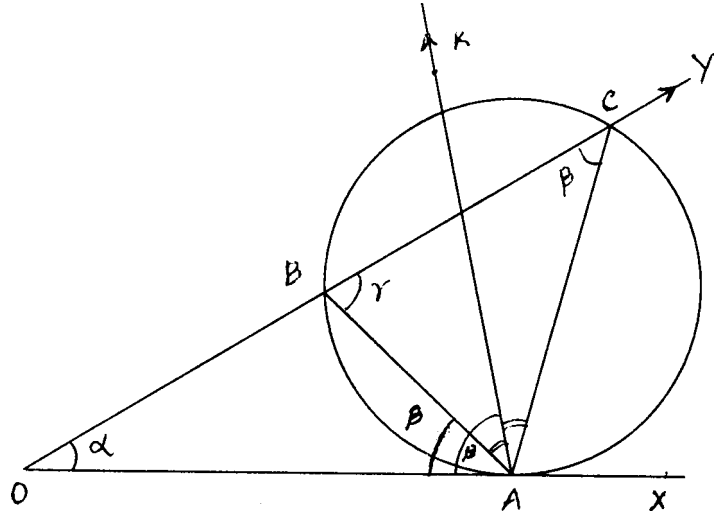
In triangles BDF and BCH, $\angle BDF = \angle BCD = 60°$ and $BD = BC, DF = CH$. Therefore the triangles are

congruent and $BF = BH$.

$\angle FBH = \angle FBD + \angle DBH = \angle HBC + \angle DBH = \angle DBC = 60°$. Hence FBH is an equilateral triangle.

4. Suppose the angle formed by the two rays OX and OY is the acute angle α and A is a given point on the ray OX. Consider all circles touching OX at A and intersecting OY at B, C. Show that the incentres of all triangles ABC lie on the same straight line.

Solution: Let AK be the internal bisector of the angle BAC made by any allowable circle. We have $\angle OAB = \angle ACB = \beta$, say (angle in the alternate segment).



$$\begin{aligned}\angle OAK &= \angle OAB + \angle BAK \\ &= \beta + \frac{1}{2}(\pi - \beta - \gamma) \\ &= \beta + \frac{1}{2}(\pi - \beta - (\alpha + \beta)) \\ &= \frac{\pi}{2} - \frac{\alpha}{2} = \text{ a constant}\end{aligned}$$

This means that the line AK is a fixed line and hence the in-centres of all triangles ABC lie on this fixed line AK.

5. Find all real numbers a and b such that

$$x^2 + ax + b^2 = 0 \text{ and } x^2 + bx + a^2 = 0$$

have at least one common root.

Solution: $x^2 + ax + b^2 = 0$ and $x^2 + bx + a^2 = 0$ have a common solution α implies

$$\alpha^2 + a\alpha + b^2 = \alpha^2 + b\alpha + a^2$$

$$\therefore \quad \alpha(a - b) = a^2 - b^2 = (a - b)(a + b)$$

$$\therefore \quad a = b \text{ or } \alpha = a + b$$

So either $a = b$ or the common root is $\alpha = a + b$.

Case 1. $a = b$

In this case the two equations are identical and they have real roots iff $a^2 - 4b^2 = -3a^2 = 0$ which is possible iff $a = b = 0$.

Case 2. $\alpha = a + b$ is the common root.

In this case, we have $(a+b)^2 + a(a+b) + b^2 = 2a^2 + 3ab + 2b^2 = 0$. Since a is given to be a real number the discriminant of this quadratic in a must be positive. Hence, $9b^2 - 16b^2 = -7b^2 \geq 0$. This can happen iff $b = 0$. So, the answer to the problem is: $a = b$ or for real roots $a = b = 0$.

6. Let a, b, c be positive real numbers such that $a+b+c \geq abc$. Prove that

$$a^2 + b^2 + c^2 \geq \sqrt{3}abc$$

Solution: Assume that $a^2 + b^2 + c^2 < \sqrt{3}abc$. Then by the AM-GM inequality, $3\sqrt[3]{a^2b^2c^2} \leq a^2 + b^2 + c^2 < \sqrt{3}abc$. It follows that $abc > 3\sqrt{3}$ $(*)$

Also, by the given condition on a, b, c ,

$$\frac{a^2b^2c^2}{3} \leq \frac{(a+b+c)^2}{3} \leq a^2 + b^2 + c^2 < \sqrt{3}abc$$

$$\Rightarrow \qquad abc < 3\sqrt{3} \qquad (**)$$

a contradiction to $(*)$. Therefore $a^2 + b^2 + c^2 \geq \sqrt{3}abc$.

7. A school has 281 boys and girls from seven countries. Suppose among any six students there are at least two who have the same age. Prove that there are five boys from the same country having the same age or there are five girls from the same country having the same age.

Solution: Among any six students there are two who have the same age. We can divide the whole group into at most

FIVE different classes such that each class contains students with the same age. Since $\left[\frac{281}{5}\right] = 56$, there is at least one class having at least 57 students having the same age. In this group of 57, since there are only seven countries, there must be at least 9 people from the same country. In this nine at least five must be of the same gender. Hence the result.

8. You are given ten segments such that each segment is longer than one cm but shorter than 55cms. Prove that you can select three sides of a triangle among the given segments.

Solution: Let the lengths of the ten segments be arranged as

$$1 \leq a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{10} < 55$$

Assume that no triangle can be constructed. Then,

$$\begin{aligned}
a_3 &\geq a_1 + a_2 > 2 \\
a_4 &\geq a_2 + a_3 > 1 + 2 = 3 \\
a_5 &\geq a_3 + a_4 > 2 + 3 = 5 \\
a_6 &\geq a_4 + a_5 > 3 + 5 = 8 \\
a_7 &\geq a_5 + a_6 > 5 + 8 = 13 \\
a_8 &\geq a_6 + a_7 > 8 + 13 = 21 \\
a_9 &\geq a_7 + a_8 > 13 + 21 = 34 \\
a_{10} &\geq a_8 + a_9 > 21 + 34 = 55
\end{aligned}$$

Hence $a_{10} > 55$ which is a contradiction to our assumption.

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AMTI

INTER LEVEL : FINAL - 2001

Instructions:

1. *Answer as many questions as possible*
2. *Elegant solutions will receive extra credits*
3. *Maximum time allotted is 3 Hrs.*

1. Let $n = 123456789101112131415\cdots979899100101$. Find the first three digits of $\sqrt{n}$. Compute the sum of the digits of the number n; show that $\sqrt{n}$ is irrational.

Solution: We are given
$n = 123456789101112131415\cdots979899100101$.

(i) n has $9 + (90 \times 2) + 6 = 195$ digits. Applying the usual algorithm for finding the square root, the first three digits are all equal to 1.

(ii) The sum of the digits $n = (23)\times1+(20)\times2+(20)\times3+(20)\times4+(20)\times5+(20)\times6+(20)\times7+(20)\times8+(20)\times9 = 23 + (44) \times (20) = 23 + 880 = 903$.

(iii) Now the sum of the digits is 903 which implies that 3 divides n but 9 does not divide n. Hence n is not a perfect square and $\sqrt{n}$ is irrational.

2. In an arithmetic progression of integers, one of the terms is the square of a natural number. Prove that there are infinitely many terms in the arithmetic progression which are squares of natural numbers.

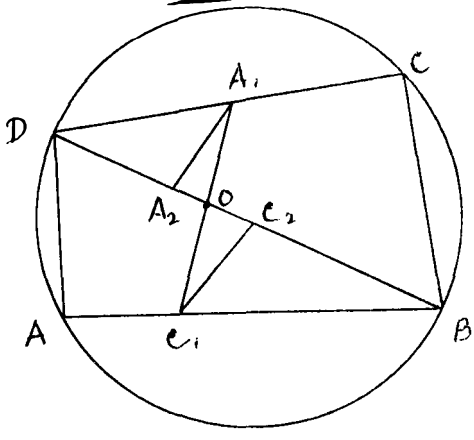
Solution: Let the arithmetic progression be $a_1, a_2, \cdots, a_n, \cdots$ where the common difference is d. Since all the terms are integers we must have d also as an integer. Suppose $a_n = m^2$ for some $m \in Z$; then $(m + kd)^2 = m^2 + d(2mk + k^2 d)$ is the term a_j where $j = n + (2mk + k^2 d) = j(n)$ (say). Then the terms $a_n, a_{j(n)}, a_{j(j(n))} \cdots$ are all perfect squares in the given $A.P.$

3. Let $ABCD$ be a cyclic quadrilateral. Points C_1 and A_1 are marked on the rays BA and DC respectively, so that $DA = DA_1$ and $BC = BC_1$. Prove that the diagonal BD intersects the segment A_1C_1 at its midpoint.

Solution: Suppose O is the point of intersection of BD with A_1C_1. Draw A_1A_2, C_1C_2 perpendicular to BD. We have

$A_1A_2 = DA_1 \sin \angle A_1DA_2 = DA \sin \angle A_1DA_2 = 2R(\sin \angle C_1BC_2) \sin \angle A_1DA_2$

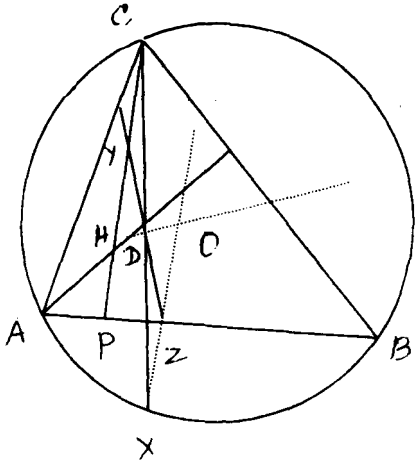
$C_1C_2 = BC_1 \sin \angle C_1BC_2 = BC \sin \angle C_1BC_2 = 2R(\sin \angle A_1DA_2) \sin \angle C_1BC_2$.



Hence $A_1A_2 = C_1C_2$. If A_1C_1 is perpendicular to BD then A_2 and C_2 coincide with the midpoint of A_1C_1. Otherwise the triangles A_1OA_2 and C_1OC_2 are congruent and hence $A_1O = C_1O$ as required.

4. Let H be the orthocentre of an acute angled triangle ABC in which side $AC \neq$ side BC. Suppose the line joining the midpoints of the segments AB and HC intersects the angular bisector of $\angle ACB$ in point D. If the centre of the circumcircle of ABC lies on HD find the measure of $\angle ACB$.

Solution: O is the cicum-centre of triangle ABC and Y, Z are the midpoints of CH and AB respectively. OZ meets the circle ABC at X. Hence X is the midpoint of the arc AXB. This means that CX is the bisector of $\angle ACB$. The straight lines HO, CX and YZ meet at D. Therefore, $\triangle HYD$ is similar to $\triangle OZD$ and $\triangle CYD$ is similar to $\triangle XZD$.



This implies that $\frac{|YD|}{|ZD|} = \frac{|YC|}{|ZX|}$ and $\frac{|YD|}{|ZD|} = \frac{|HY|}{|OZ|}$. This in turn implies that $|OZ| = |ZX|$. Again, this implies that $\angle AOB = \angle AXB = 180 - \angle ACB$. But $\angle AOB = 2\angle ACB$. Thus we get $\angle ACB = 60°$.

5. Let k and $n_1 < n_2 < \cdots < n_k$ be odd positive integers. Show that

$$n_1^2 - n_2^2 + n_3^2 - n_4^2 + \cdots + n_k^2 \geq 2k^2 - 1$$

Proof. The case $k = 1$ is clearly true! We prove the result by induction. Assume that

$$n_1^2 - n_2^2 + n_3^2 - n_4^2 + \cdots + n_k^2 \geq 2k^2 - 1 \qquad (*)$$

Now, we would like to prove $n_1^2 - n_2^2 + n_3^2 - n_4^2 + \cdots + n_k^2 - n_{k+1}^2 + n_{k+2}^2 \geq 2(k+2)^2 - 1 = 2k^2 - 1 + 8k + 8$.

From our assumption $(*)$, we see that it is enough to prove that

$$n_{k+2}^2 - n_{k+1}^2 \geq 8k + 8 \qquad (**)$$

Now, $(n_{k+2} - n_{k+1}) \geq 2$ and $n_k \geq 2k - 1$ and therefore, we get

$(n_{k+2} - n_{k+1})(n_{k+2} + n_{k+1})$

$$= n_{k+2}^2 - n_{k+1}^2 \geq 2((2k+1) + (2k+3)) = 8k + 8.$$

This proves $(**)$ and hence the result is true for $k+1$. The result follow by the principle of mathematical induction.

6. Let a and b be complex non-zero numbers and z_1, z_2 the roots of the polynomial

$$X^2 + aX + b.$$

Show that

$$|z_1 + z_2| = |z_1| + |z_2|$$

if and only if there exists a real number $\lambda \geq 4$ such that $a^2 = \lambda b$.

Proof. We note that for any two complex numbers z_1 and z_2

$$|z_1 + z_2| = |z_1| + |z_2| \text{ iff } \ni \text{ a real number } r > 0$$

such that $z_1 = rz_2$.

Therefore when $|z_1 + z_2| = |z_1| + |z_2|$ implies that

$$\frac{a^2}{b} = \frac{(z_1 + z_2)^2}{z_1 z_2} = \frac{(r+1)^2}{r} = \lambda,$$

say where $\lambda \geq 4$. $\therefore$ $a^2 = \lambda b, \lambda \geq 4$. Conversely, let $a^2 = \lambda b, \lambda \geq 4$; then the discriminant of the quadratic $X^2 + aX + b$ is $\triangle = a^2 - 4b = a^2(\lambda - 4)/\lambda$. Therefore the roots are

$$z_1, z_2 = \frac{a}{2}\left(-1 \pm \sqrt{\frac{\lambda - 4}{\lambda}}\right)$$

$$\therefore \quad \frac{z_1}{z_2} = \frac{\sqrt{\lambda} \pm \sqrt{\lambda - 4}}{\sqrt{\lambda} \mp \sqrt{\lambda - 4}} = \frac{\left(\sqrt{\lambda} \pm \sqrt{\lambda - 4}\right)^2}{4} = r > 0.$$

7. In the AMTI training session, in order to get best results, the coordinator followed the following rule: Everyday

the number of problems solved is at most ten; but if on a particular day the number of problems solved is greater than seven, then in the next two days the number of problems solved is at most five per day. Find the maximum number of problems that can be solved in a training camp of 25 days.

Proof. Let n_i be the number of problems solved on the i^{th} day for $i = 1, 2, 3, \cdots, 25$. Let A be the largest possible number of problems solved in the camp by the coordinator during the 25 days and $A = n_1 + n_2 + \cdots + n_{25}$.

Claim: $n_i \leq 7$ for $1 \leq i \leq 23$.

Suppose there exists j such that $1 \leq j \leq 23$ with $n_j > 7$. Then by our condition we must have $n_{j+1} = n_{j+2} = 5$ and $n_j + n_{j+1} + n_{j+2} \leq 10 + 5 + 5 = 20$. Whereas, if $n_j = n_{j+1} = n_{j+2} = 7$ the corresponding sum $n_j + n_{j+1} + n_{j+2} = 21 > 20$ implying that A cannot be the largest number of problems solved. Hence the claim.

This means that $A \leq 7(23) + 10 + 7 = 161 + 17 = 178$ and this maximum is attained when $n_1 = n_2 = n_3 = \cdots = n_{24} = 7$ and $n_{25} = 10$. Therefore the answer to the problem is 178.

8. In a Mathematical Olympiad four problems were given and each problem is worth four marks. Any contestant can score 0,1,2,3 or 4 marks in each problem. If no two contestants obtained the same marks in more than one problem, find the maximum possible number of contestants in the olympiad.

Proof. The possible scores that can be got in any problem are 0,1,2,3,4. These are FIVE in number. Therefore at most Five participants could have scored the same number of points in the first problem. Otherwise we must have SIX or more different scores for the second problem; which is impossible. This means that at most 5 contestants scored 0 point, at most 5 scored 1 point, at most 5 scored 2 points, at most 5 scored 3 points an at most 5 scored 4 points in the first problem. This implies that the number of contestants is at most 25. We give below a configuration where the maximum 25 is attained. Hence the answer to the problem is 25.

Contestants	Problem 1	Problem 2	Problem 3	Problem 4
1	0	0	0	0
2	0	1	1	1
3	0	2	2	2
4	0	3	3	3
5	0	4	4	4
6	1	1	2	3
7	1	2	3	4
8	1	3	4	0
9	1	4	0	1
10	1	0	1	2
11	2	2	4	1
12	2	3	0	2
13	2	4	1	3
14	2	0	2	4
15	2	1	3	0
16	3	3	1	4
17	3	4	2	0
18	3	0	3	1
19	3	1	4	2
20	3	2	0	3
21	4	4	3	2
22	4	0	4	3
23	4	1	0	4
24	4	2	1	0
25	4	3	2	1

SOLUTIONS COMPILED BY: MS. R. VIJAYALAKSHMI, 36,Tyagaraya Grammani St., T.Nagar, Chennai-17 and
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AMTI

SENIOR LEVEL : FINAL - 2001

Note:

1. *Answer as many questions as possible*
2. *Elegant solutions will receive extra credits*
3. *Maximum time allotted is 3 Hrs.*

1. Show that if a, b, c are complex numbers such that

$$(a+b)(a-b) = b-c$$
$$(b+c)(b-c) = c-a$$
$$(c+a)(c-a) = a-b$$

then a, b, c are real numbers.

Solution:

$$(a+b)(a-b) = b-c \tag{1}$$

$$(b+c)(b-c) = c-a \tag{2}$$

$$(c+a)(c-a) = a-b \tag{3}$$

(1) - (2) gives $(a+b)(a-b) = b-c$ (4)

(2) - (3) gives $(b+c)(b-c) = c-a$ (5)

(3) - (1) gives $(c+a)(c-a) = a-b$ (6)

Hence we observe that if two of the three numbers a, b, c are equal then all the three are equal - in which case $a = b = c = 0$ or $a = b = c = 1/4$. Assume now that the numbers are distinct. Then (4), (5) and (6) give $(a+b)(b+c)(c+a) = 1$. This , along with (1), (2) and (3) gives $b(b+c) = c(c+a) = a(a+b) = 1$. Again, if one of the numbers is real then the other two have to be real. If a, b, c are all not real, then we

may assume that a, b are in the upper half plane or a, b are in the lower half plane. Therefore, a and $a+b$ are both in the upper half plane or $a, a+b$ are both in the lower half plane. But then $a(a+b)$ cannot be equal to 1. Thus a, b, c have to be all real numbers.

2. Find all positive integers a, b, c satisfying the equality $a! + 2001b! = c!$

Solution: Clearly we must have $c > a$ and $c > b$. If $b > a$, then we get $1 = 0 \mod b$ on dividing $a! + 2001b! = c!$ by $a!$. This makes $b = 1$ and $b > a$ is impossible since a, b, c are all positive integers.

Let now $a > b$. In this case, again by dividing by $a!$, we get

$$2001 \text{ is divisible by } a(a-1)(a-2)\cdots(b+1)$$

which is impossible unless $a = b + 1$ (note that 2001 is odd). Assume now that $a = b + 1$. This means that 2001 is divisible by a. We check that none of the divisors 3, 23, 29, $3 \times 23, 3 \times 29, 23 \times 29$ and 2001 of 2001 gives a solution of our equati

$3! + 2001 \times 2! = 4008 \neq c!$	$23! + 2001 \times 22! = 88 \times 23! \neq c!$
$29! + 2001 \times 28! = 70 \times 29! \neq c!$	$69! + 2001 \times 68! = 30 \times 69! \neq c!$
$87! + 2001 \times 86! = 24 \times 87! \neq c!$	$667! + 2001 \times 666! = 4 \times 667! \neq c!$
$2001! + 2001 \times 2000! = 2 \times 2001! \neq c!.$	

In the case $a = b$,we get $a = b = 2001$ and $c = 2002$ as the only solution.

3. Prove that

$$\frac{x(y+z)}{(x-y)(x-z)} + \frac{y(z+x)}{(y-z)(y-x)} + \frac{z(x+y)}{(z-x)(z-y)} = -1.$$

Solution: This is straightforward algebraic simplification.

$$\frac{x(y+z)}{(x-y)(x-z)} + \frac{y(z+x)}{(y-z)(y-x)} + \frac{z(x+y)}{(z-x)(z-y)} = -1.$$

$$= \frac{x(y+z)(z-y)(z+x)(x-z) + z(x+y)(y-x)}{(x-y)(y-z)(z-x)}$$

$$= \frac{x(z-y)(z+x) + x^2(y-z) - yz(z-y)}{(x-y)(y-z)(z-x)}$$

$$= \frac{(y-z)(x^2 - x(z+y) + yz)}{(x-y)(y-z)(z-x)}$$

$$= \frac{(y-z)(x-z(x-y)}{(x-y)(y-z)(z-x)} = -1.$$

4. Find all polynomials with real coefficients $P(x)$ such that

Solution: Let the degree of $P(x)$ be n. Then $P(2x-1) = 2^n P(x) + r(x)$ where degree $r(x) < n$ or $r(x) = 0$. Substituting in the given equation, we get

$$P(x)(2^n P(x^2) + r(x^2)) = P(x^2) \cdot (2^n P(x) + r(x))$$

Hence

$$P(x) \cdot r(x^2) = P(x^2) \cdot r(x).$$

If $r(x) \neq 0$, let its degree be m. Then the above equation implies that

$$n + 2m = m + 2n.$$

In other words we must have $m = n$ which is a contradiction as deg $r(x) <$ deg$P(x)$. Therefore $r(x) = 0$ or $P(2x-1) = 2^n P(x)$. This in turn implies that $P(2x+1) = 2^n P(x+1)$. f $f(x) = P(x+1)$ then $f(2x) = 2^n f(x)$. This means that if $(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ then $2^k a_k = 2^n a_k$ ior $k = 0, 1, 2, \cdots, n-1$ or $a_k = 0$ for $0 \leq k \leq n-1$. Therefore $f(x) = a_n x^n$ or $P(x) = \lambda(x-1)^n$ where λ is real and $n \geq 0$ or $P(x) = c = a$ constant. One can check that such polynomials $P(x)$ satisfy the given equation.

5. Prove that there is a constant C such that if $p(x)$ is a polynomial of degree 1999 then $p(0) \leq C\int_{-1}^{1} |p(x)|dx$.

Solution: Let $t_1, t_2, \cdots, t_{1999}$ be the roots of $p(x)$ in the complex plane. Consider the open disks

$$B_k = \{z| \ |z - t_k| < \epsilon \text{ where } 0 < \epsilon < (1/3998)\}$$

for $k = 1, 2, \cdots, 1999$. Each B_k determines a sub interval J_k of $[-1/2, 1/2]$ of length less than $2\epsilon < (1/1999)$. Removing these 1999 sub intervals from $[-1/2, 1/2]$ we get 2000 or fewer sub intervals I_k of $[-1/2, 1/2]$ whose total length is less than $1 - 1999(2\epsilon) = 1 - 3998\epsilon$. Choose one of these subintervals I_k such that length of it is greater than $\frac{1-3998\epsilon}{2000}$. Call such an interval I and let the length of I be δ. Now,

$$\frac{|p(x)|}{|p(0)|} = \prod_{k=1}^{1999} \frac{|x - t_k|}{|t_k|}$$

If x is in I, then $|x - t_k| \geq \epsilon$ for each k. If $|t_k| \leq 1$, then $\frac{|x-t_k|}{|t_k|} \geq \epsilon$ for x in I. On the other hand if $|t_k| > 1$, then

$$\frac{|x - t_k|}{|t_k|} = \left|1 - \frac{x}{t_k}\right| \geq 1 - \left|\frac{x}{t_k}\right| \geq \frac{1}{2} \geq \epsilon.$$

Thus in either case,

$$\frac{|p(x)|}{|p(0)|} = \prod_{k=1}^{1999} \frac{|x - t_k|}{|t_k|} \geq \epsilon^{1999}.$$

Therefore,

$$\int_I \frac{|p(x)|}{|p(0)|} dx \geq \delta\epsilon^{1999} = C$$

which is a constant independent of $p(x)$. Thus,

$$p(0) \leq C \int_{-1}^{1} |p(x)| dx.$$

6. The sequence (a_n) is defined by $a_1 = 1$, $a_2 = 2, a_3 = 24$ and for $n \geq 4$

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}$$

For all n show that a_n is an integral multiple of n.

Solution: We write

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}} \quad \text{as}$$

$$\frac{a_n}{a_{n-1}} = \frac{6a_{n-1}}{a_{n-2}} - \frac{8a_{n-2}}{a_{n-3}}$$

If we define $b_n = \frac{a_n}{a_{n-1}}$ then we have $b_2 = 2$ and $b_3 = 12$ and $b_n = 6b_{n-1} - 8b_{n-2}$. Solving,we get

$$b_n = \frac{1}{4} \times 4^n - \frac{1}{2} \times 2^n = 2^{n-1}(2^{n-1} - 1) \text{ for } n \geq 4$$

$$\therefore \quad \frac{a_2}{a_1} \cdot \frac{a_3}{a_2}, \cdots \frac{a_n}{a_{n-1}} = b_2 \cdot b_3 \cdots b_n$$

$$\text{or } a_n = 2 \times 12 \times \prod_{j=3}^{n-1} 2^j(2^j - 1),$$

since $a_1 = 1, b_2 = 2$ and $b_3 = 12$

$$\text{or } a_n = 2 \times 2^2 \times \cdots \times 2^{n-1} \prod_{j=1}^{n-1} (2^j - 1)$$

$$= 2^{\frac{n(n-1)}{2}} \prod_{j=1}^{n-1} (2^j - 1).$$

This gives

$$a_n = 2^{n(n-1)/2} \prod_{j=1}^{n-1} (2^j - 1)$$

If $n = 2^k m$ where m is odd, then $k < n < n(n-1)/2$ and there exists $\varphi(m) = j \leq (m-1)$ such that m divides $2^j - 1$. This means that n divides a_n.

7. Let $p(x)$ be a polynomial that is non-negative for all real x. Prove that for some k there exist polynomials $f_1, f_2, \cdots, f_k$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2$$

Solution: Write $p(x) = f(x)g(x)$ where $f(x)$ has all its roots real and $g(x)$ has all its roots imaginary. Then since $p(x) \geq 0$, each root of $f(x)$ must have multiplicity even, otherwise $p(x)$ changes sign at those roots. Thus $f(x) = (F(x))^2$ for some polynomial $F(x)$. Now $g(x)$ can be written as $g(x) = \prod_j (x-a_j)(x-\bar{a}_j)$. If $\prod_j (x-a_j) = u(x)+iv(x)$ then we have

$$p(x) = f(x)g(x) = (F(x))^2((u(x) + iv(x))(u(x) - iv(x)))$$

$$= (F(x))^2((u(x))^2 + (v(x))^2) = \sum_{j=1}^{k} (f_j(x))^2$$

8. Consider all the non empty subsets of $\{1, 2, 3, \cdots, 2001\}$. For every such subset find the product of the reciprocals of each of its elements. Find the sum of all these products.

Solution: This is proved by induction. Consider all nonempty subsets of $\{1, 2, 3, \cdots, n\}$. Let S_n be the sum required. We claim that $S_n = n$ for all n. The result is true for $n = 1$. Assume the result for n and consider the set $(1, 2, 3, \cdots, n+1\}$. For this set

$$S_{n-1} = \frac{1}{n+1}(S_n + 1) + S_n = \frac{1}{n+1}(n+1) + n = n + 1.$$

Hence the result follows by the principle of mathematical induction.

9. A closed disk of radius 1 contains seven points with mutual distances ≥ 1. Prove that the centre of the disk is one of the seven points.

Solution: Suppose the centre O is not any one of the seven points. Then there are two of the seven points A and B such that AB subtends an angle less than 60 degrees at O. Then we have

$$\begin{aligned} AB^2 &= OA^2 + OB^2 - 2OA\,OB\cos\angle AOB \\ &< 1 + 1 - 2\left(\frac{1}{2}\right) = 1. \end{aligned}$$

This is a contradiction to our assumption on the seven points.

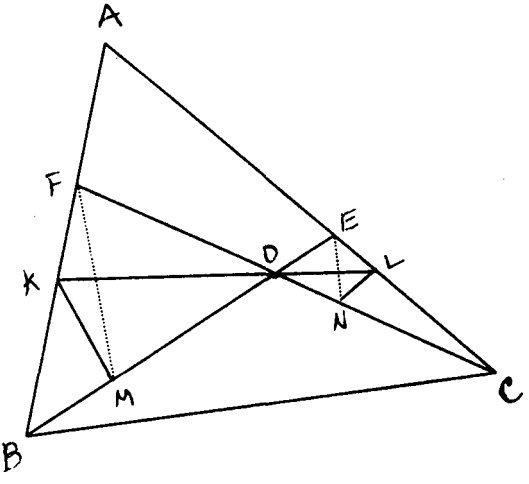
SOLUTIONS COMPILED BY: PROF. K.N.RANGANATHAN, Professor of Mathematics, Sri RKM Vivekananda College, Chennai-600004.

REGIONAL MATHEMATICAL OLYMPIAD-2001
(Sponsored by NBHM)

Time: 3 hours December 02,2001

1. Let BE and CF be the altitudes of an acute triangle ABC, with E on AC and F on AB. Let O be the point of intersection of BE and CF. Take any line KL through O with K on AB and L on AC. Suppose M and N are located on BE and CF respectively, such that KM is perpendicular to BE and LN is perpendicular to CF. Prove that FM is parallel to EN.

Solution.



In the above figure BE is perpendicular to AC, CF is perpendicular to AB, KM is perpendicular to BE, and LN is perpendicular to CF. Now as $\angle OMK + \angle OFK = 90° + 90° = 180°$, $OFKM$ is a cyclic quadrilateral. (1)

Similarly as $\angle OEL + \angle ONL = 90° + 90° = 180°, ONLE$ is also a cyclic quadrilateral. (2)

Again, $\angle NFM = \angle OFM = \angle OKM$ (3)

as $\angle OFM$ and $\angle OKM$ are angles subtended by the same arc OM of the circle through the points O, F, K, M using (1).

Now,

$\angle OKM = 90° - \angle KOM$, as $\triangle OMK$ is right angled at M
$= 90° - \angle LOE$, as $\angle LOE$ is opposite to $\angle KOM$
$= \angle OLE$, as $\triangle OLE$ is right angled at E
$= \angle ENO$, (angles in the same segment)
$= \angle ENF$ i.e., $\angle OKM = \angle ENF$ (4)

From (3) and (4) we get

$$\angle NFM = \angle OKM = \angle ENO$$

Hence FM is parallel to EN.

2. Find all primes p and q such that $p^2 + 7pq + q^2$ is the square of an integer.

Solution.

(Communicated by Dr.C.R.Parameshachar, Mathematics Olympiad Cell, I.I.Sc., Bangalore -12)

We must have $p^2 + 7pq + q^2 = a^2$ for some $a \subset \mathbb{N}$. So $a^2 - (p+1)^2 = 5pq$ i.e., $(a + p + q)(a - p - q) = 5pq$. Now,

as $a+p+q>5$ or p or q, we have $a+p+q=5p$ or $5q$ or pq

(a) If

$$a+p+q=5p, \tag{1}$$

then

$$a-p-q=q. \tag{2}$$

(1) and (2) give

$$2(p+q)=5p-q \Rightarrow 2p+2q=5p-q \Rightarrow 3p=3q \Rightarrow p=q.$$

The case $a+p+q=5q$ is similar.

(b) If $a+p+q=pq$, then $a-p-q=5$ and so $2p+2q=(a+p+q)-(a-p-q)=pq-5$ and so $pq-2p-2q+4=5+4=9$; i.e., $(p-q)(q-2)=9=1\times 9=9\times 1=3\times 4$.

Taking all possible cases $(p-2,q-2)=(1,9)$ or $(9,1)$ or $(3,3)$. So we get $(p,q)=(3,11)$ or (11,3),(5,5). Of these the last pair (5,5) is already included in case (a). Hence the solution set (p,q) is

$$\{(p,p)|p, \text{ a prime}\}\cup\{(3,11),(11,3)\}.$$

3. Find the number of positive integers x which satisfy the condition

$$\left[\frac{x}{99}\right]=\left[\frac{x}{101}\right].$$

(Here $[z]$ denotes, for any real z, the largest integer not exceeding z; e.g. $[\frac{7}{4}]=1$.)

Solution. Let us list the positive integral values of x satisfying the condition $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right]$ and count the number of such values of x.

Positive integral value of x satisfying the given condition	The value of $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right]$	The number of values of x
1 to 98	0	98
101 to 197	1	97
202 to 296	2	95
303 to 395	3	93
404 to 494	4	91
...	⋮	⋮
4848 to 4850	48	3
4949 only	49	1

Hence total number of such values of x =

$$\begin{aligned} &= 98 + 97 + 95 + 93 + 91 + \cdots + 3 + 1 \\ &= (1 + 3 + 5 + \cdots + 95 + 97) + 98 \\ &= \left[\frac{49}{2}(1 + 97)\right] + 98 = \frac{49 \times 98}{2} + 98 \\ &= 2499. \end{aligned}$$

For other values of x, $\left[\frac{x}{99}\right] \neq \left[\frac{x}{101}\right]$.

4. Consider an $n \times n$ array of numbers:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

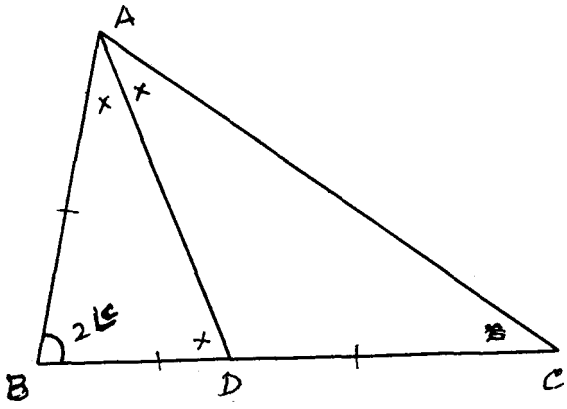
Suppose each row consists of the n numbers $1,2,3,\cdots,n$ in some order and $a_{ij} = a_{ji}$ for $i = 1,2,\cdots,n$ and $j = 1,2,\cdots,n$. If n is odd, prove that the numbers $a_{11},a_{22},a_{33},\cdots,a_{nn}$, are $1,2,3,\cdots,n$ in some order.

Solution. As there are n rows in the array and each row consists of the n numbers $1,2,3,\cdots,n$, there is a total of n^2 numbers and each of the numbers $1,2,\cdots,n$ occurs n times in the array. Also as $a_{ij} = a_{ji}$ for $1 \leq i,j \leq n$, $(i,j, \in \mathbb{N})$ and n is odd $(= 2k+1$, say), the array is symmetric about the diagonal $D = \{a_{11},a_{22},a_{33},\cdots,a_{nn}\}$ of the array. So the number of times any specific number of the set $S \equiv \{1,2,3,\cdots,n\}$ will occur on both sides of the diagonal D, but <u>not on D</u> is even (say $2m, m \leq k, m \in \mathbb{N}$). So this specific number will occur an odd number of times $(= 2k+1-2m)$ in D. This holds for each of the numbers $1,2,3,\cdots,n$. i.e., each number of $1,2,3,\cdots,n$ occurs at least once in D.

Now D has n numbers $a_{11},a_{22},a_{33},\cdots,a_{nn}$, as its elements and also each of the numbers $1,2,3,\cdots,n$ occurs at least once in D. Hence the numbers $a_{11},a_{22},a_{33},\cdots,a_{nn}$, are the numbers $1,2,3,\cdots,n$ in some order.

5. In a triangle ABC, D is a point on BC such that AD is the internal bisector of $\angle A$. Suppose $\angle B = 2\angle C$ and $CD = AB$. Prove that $\angle A = 72°$.

Solution. AD is the internal bisector of $\angle A$



$$\Rightarrow \quad \angle BAD = \angle OAD$$

$$\Rightarrow \quad \frac{BD}{CD} = \frac{AB}{AC} = \frac{c}{b}$$

$$\Rightarrow \quad \frac{BD + CD}{CD} = \frac{b + c}{b}$$

$$\Rightarrow \quad \frac{BC}{CD} = \frac{b + c}{b}$$

$$\Rightarrow \quad \frac{a}{CD} = \frac{b + c}{b}$$

$$\Rightarrow \quad \frac{a}{AB} = \frac{b + c}{b} \text{ as } CD = AB$$

$$\Rightarrow \quad \frac{a}{c} = \frac{b + c}{b}$$

$$\Rightarrow \quad ab = bc + c^2 \qquad (1)$$

Also, given that $\angle B = 2\angle C$

$$\Rightarrow \quad \angle B - \angle C = \angle C \Rightarrow \sin(B - C) = \sin C$$

$$\Rightarrow \quad \sin(B - C) \cdot \sin(B + C) = \sin C \cdot \sin(B + C)$$

$$\Rightarrow \quad \sin(B - C)\sin(B + C) = \sin C \cdot \sin A \qquad (2)$$

for $\sin(B+C) = \sin(180° - A) = \sin A$. So from (2), $\sin^2 B - \sin^2 C = \sin C \cdot \sin A$

$$\Rightarrow \quad b^2 - c^2 = ca$$

$$\Rightarrow \quad b^2 = c^2 + ca \qquad (3)$$

(1) and (3) give $ab - b^2 = bc - ca$

$$\Rightarrow \quad b(a-b) = -c(a-b)$$

$$\Rightarrow \quad (b+c)(a-b) = 0$$

$$\Rightarrow \quad a - b = 0$$

for $b + c > 0, b$ and c being sides of $\triangle ABC$

$$\Rightarrow \quad a = b \qquad (4)$$

(4) gives $\angle A = \angle B = 2\angle C$

$$\Rightarrow 180^\circ = \angle A + \angle B + \angle C = 2\angle C + 2\angle C + \angle C = 5\angle C$$

$$\Rightarrow \quad \angle C = \frac{180^\circ}{5} = 36^\circ$$

Hence $\angle A = 2\angle C = 2 \times 36^\circ = 72^\circ$.

6. If x, y, z are the sides of a triangle, then prove that

$$|x^2(y-z) + y^2(z-x) + z^2(x-y)| < xyz.$$

Solution.

$$x^2(y-z) + y^2(z-x) + z^2(x-y)| < xyz$$

$$= \quad x^2(y-z) + y^2z - y^2x + z^2x - z^2y$$

$$= \quad x^2(y-z) + (y^2z - z^2y) + (z^2x - y^2x$$

$$= \quad x^2(y-z) + yz(y-z) + x(z+y)(z-y)$$

$$= \quad (y-z)[x^2 - xy - xz + yz]$$

$$= \quad (y-z)[x(x-y) - z(x-y)$$

$$= \quad (y-z)(x-y)(x-z)$$

Now, as x, y, z are the sides of a triangle and in a triangle the difference between any two sides is less than the third side, we have

$$|y-z| < x,\ |x-y| < z, |x-z| < y$$

$$\Rightarrow \quad |y-z|\ |x-y|\ |x-z| < xyz$$

$$\begin{aligned} \text{Hence} \qquad & |x^2(y-z) + y^2(z-x) + z^2(x-y)| \\ = \quad & |(y-z)\ (x-y)(x-z)| \\ = \quad & |y-z|\ |x-y|\ |x-z| < xyz \end{aligned}$$

7. Prove that the product of the first 1000 positive even integers differs from the product of the first 1000 positive odd integers by a multiple of 2001.

Solution. The expression
$(2 \cdot 4 \cdot 6 \cdot \cdots 1000 \text{ factors}) - (1 \cdot 3 \cdot 5 \cdot \cdots 1000 \text{ factors})$
$= (2 \cdot 4 \cdot 6 \cdot \cdots 2000) - (1 \cdot 3 \cdot 5 \cdot \cdots 1999)$

Now, $2001 = 3 \times 23 \times 29$ and $3, 23, 29$ are coprimes two by two. As 6, 46 and 58 are divisible by 3, 23, 29 respectively and occur as factors in $2 \cdot 4 \cdot 6 \cdot \cdots 2000$, the product $2 \cdot 4 \cdot 6 \cdot \cdots 2000$ is divisible by 2001. Also, as 3,23 and 29 occur as factors in the product $1 \cdot 3 \cdot 5 \cdot \cdots 1999$, the product $1 \cdot 3 \cdot 5 \cdot \cdots 1999$ is divisible by 2001. Hence their difference $(2 \cdot 4 \cdot 6 \cdot \cdots 2000) - (1 \cdot 3 \cdot 5 \cdot \cdots 1999)$ is divisible by 2001.

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SOLUTIONS TO INMO 2002

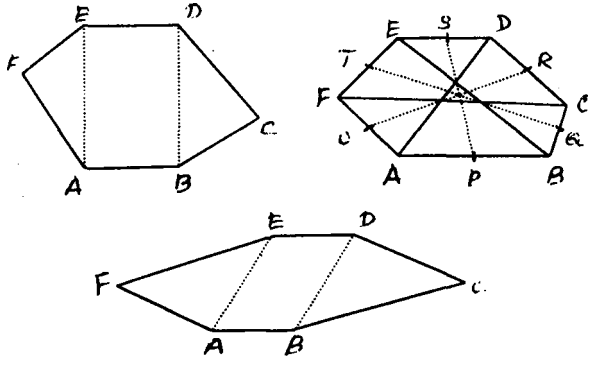
1. For a convex hexagon $ABCDEF$ consider the following six statements.

(a_1) AB is parallel to DE	(a_2) $AE = BD$
(b_1) BC is parallel to EF	(b_2) $BF = CE$
(c_1) CD is parallel to FA	(c_2) $CA = DF$

(a) Show that if all the six statements are true, then the hexagon is cyclic.

(b) Prove that, in fact, any five of the six statements also imply that the hexagon is cyclic.

Solution. The problem should have an extra condition, namely, 'the opposite sides are unequal'. Otherwise, we have the following counter example; take $AB = DE$ which makes $ABDE$ a parallelogram. The only parallelograms which are concyclic are the rectangles.



Assume now that in the given hexagon $ABCDEF$ the opposite sides are unequal; and all the six conditions hold good. Let P, Q, R, S, T and U be the mid-points of the sides AB, BC, CD, DE, EF and FA respectively.

Case (1): AD, BE and CF meet at a point O. We have $ABDE, BCEF$ and $CDFA$ as isosceles trapeziums. Hence O must be on the perpendicular bisectors of AB, BC, CD, DE, EF and FA. This means that $OA = OB = OC = OD = OE = OF$ and $ABCDEF$ is a cyclic hexagon with O as its centre.

Case (2): Assume that AD, BE and CF are not concurrent. Then they form a triangle XYZ as shown in the figure. Now, PX extended, RY extended and TZ extended should be the internal angular bisectors of the triangle XYZ. Now, PX extended, RY extended and TZ extended should be the internal angular bisectors of the triangle XYZ. Therefore they meet at the in-centre I of the triangle XYZ. As seen earlier, I lies on the perpendicular bisector of each of the sides of the hexagon. Thus A, B, C, D, E and F lie on a circle with centre I.

Second Part: We are going to prove that any five of the six statements are enough to ensure that the hexagon is cyclic. Suppose that AB is parallel to DE, $AE = BD, BC$ is parallel to $EF, BF = CF$ and CE and CD is parallel to AF. Then we have $AD = BE = CF$. Now, $\triangle YCD$ is

similar to $\triangle YFA$. Therefore

$$\frac{FY}{AY} = \frac{CY}{YD} = \frac{FY + CY}{AY + YD} = \frac{CF}{AD} = 1$$

Therefore, $FY = AY$ and $CY = YD$.

This implies that the triangles CYA and DYF are congruent and hence $AAC = DF$. This means that all the six conditions hold good and the hexagon is cyclic.

Suppose that AB is parallel to $DE, AE = BD, BC$ is parallel to $EF, BF = CE$ and $AC = DF$. Now, we shall prove that the sixth condition, namely, CD parallel to FA is also satisfied. Again, we have, $AD = BE = CF$. Therefore $\triangle FDC = \triangle ACD$. This means that $\angle ADC = \angle FCD$. Similarly $\angle CFA = \angle DAF$. These observations imply that CD is parallel to FA.

Thus, whenever any five of the six statements are true then the sixth statement is also true and the hexagon is cyclic.

2. Determine the least positive value taken by the expression

$$a^3 + b^3 + c^3 - 3abc$$

as abc vary over all positive integers. Find also all triples (a, b, c) for which this least value is attained.

Solution. We have,

$$X = a^3+b^3+c^3-3abc = \frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$$

Since we are interested in the least positive value of X, a, b, c are not all equal. Therefore, the integers a, b, c should satisfy

$$a+b+c \geq 1+1+2 = 4;\ (a-b)^2+(b-c)^2+(c-a)^2 \geq 0+1+1+2$$

This means that $X \geq 4$. If $a = 1, b = 1, c = 2$ we get $X = 4$. Hence the least positive value for X is 4 and it is attained when $(a, b, c) \in \{(1,1,2)(1,2,1)(2,1,1)\}$.

3. Let x, y be positive real such that $x + y = 2$. Prove that

$$x^3y^3(x^3 + y^3) \leq 2.$$

Solution. By the Arithmetic Mean – Geometric Mean inequality,

$$xy \leq \left(\frac{x+y}{2}\right)^2 = 1$$

$x^3y^3(x^3 + y^3) = (xy)^3(x + y)(x^2 - xy + y^2) = 2(xy)^3\{(x+y)^2 - 3xy\} = 2(xy)^3(4 - 3xy)$. Therefore it is enough to prove that

$$(xy)^3(4 - 3xy) \leq 1.$$

Let $xy = t$. Consider the four positive quantities $4 - 3t, t, t$. Again, by $AM - GM$ inequality, we have,

$$t^3(4 - 3t) \leq \left(\frac{4 - 3t + 3t}{4}\right)^4 = 1.$$

4. Do there exist 100 lines in the plane, no three of them concurrent, such that they intersect exactly in 202 points?

Solution. Consider a set A of 100 straight lines so that no three are concurrent. By looking at the subsets of parallel lines, we may partition the set A into a finite number of sets $A_1, A_2, A_3, \cdots, A_n$ such that

1. any two lines in A_j are parallel, for $1 \leq j \leq n$ (or else $|A_j| = 1$)
2. for $i \neq j$, the lines in A_i and A_j are not parallel.

Then the number of points of intersection is $\sum_{1 \leq j \leq l \leq n} m_j m_l$ where $|A_j| = m_j$.

The problem is reduced to finding $m_1, m_2, m_3, \cdots, m_n$ such that

$$\sum_{j=1}^{n} m_j = 100 \quad \text{and} \quad \sum_{1 \leq j \leq l \leq n} m_j m_l = 2002.$$

This gives

$$\sum_j m_j^2 = \left(\sum_j m_j \right)^2 - 2 \sum_{1 \leq i \leq j \leq n} m_i m_j$$

$$= 10000 - 2(2002) = 5996 = 77^2 + 67.$$

So, we may choose $m_1 = 77$; this makes $\sum_2^n m_j = 23, \sum_2^n m_j^2 = 67$. Now, take $m_2 = 5, m_3 = 4, m_4 = 4, m_5 = m_6 = \cdots = m_{14} = 1$. Then we note that

$$\sum_{j=1}^{14} m_j = 100 \quad \text{and} \quad \sum_{1 \leq j \leq l \leq n} m_j m_l = 2002.$$

This demonstrates the existence of 100 lines with exactly 2002 points of intersection.

5. Do there exist three distinct positive numbers a, b, c such that

$$a, b, c, d+c-a, c+a-b, a+b-c \text{ and } a+b+c$$

form a 7-term arithmetic progression in some order?

Solution. Assume that there exists such a set a, b, c forming an arithmetic progression in some order. We may assume that $a < b < c$. Then there are only two possibilities.

Case (1). $a+b-c < a < c+a-b < b < c < b+c-a < a+b+c$. In this case, if d is the common difference of the progression, then

$$c = a+b+c-2d;\ b = a+b+c-3d;\ a = a+b+c-5d$$

Adding, we get

$$a+b+c = 3(a+b+c) - 10d \Rightarrow a+b+c = 5d \Rightarrow a = 0.$$

This is a contradiction to our assumption that $a > 0$.

Case (2). $a+b-c < a < c+a-b < b < c < b+c-a < a+b+c$. In this case, we have

$$c = a+b+c-2d;\ b = a+b+c-4d;\ a = a+b+c-5d$$

Adding, we get

$$a+b+c = 3(a+b+c) - 11d \Rightarrow a+b+c = \frac{11d}{2}$$

$$\therefore a = \frac{d}{2},\ b = \frac{3d}{2},\ c = \frac{7d}{2}.$$

Also we have

$$a + b - c = (a + b + c) - 6d = \frac{11d}{2} - bd = -\frac{d}{2}.$$

$$a + b - c = (1 + 3 - 7)\frac{d}{2} = \frac{-3d}{2}.$$

This implies that $d = 0$; which again leads to a contradiction. Thus there is no such set a, b, c of positive numbers.

6. Suppose n^2 numbers $1, 2, 3, \cdots, n^2$ are arranged to form an n by n array consisting of n rows and n columns such that the numbers in each row (from left to right) and each column (from top to bottom) are in increasing order. Denote by a_{jk} the number in the j^{th} row and k^{th} column. Suppose b_j is the maximum possible number of entries that can occur as $a_{jj}, 1 \leq j \leq n$. Prove that

$$b_1 + b_2 + b_3 + \cdots + b_n = \frac{n}{3}(n^2 - 3n + 5).$$

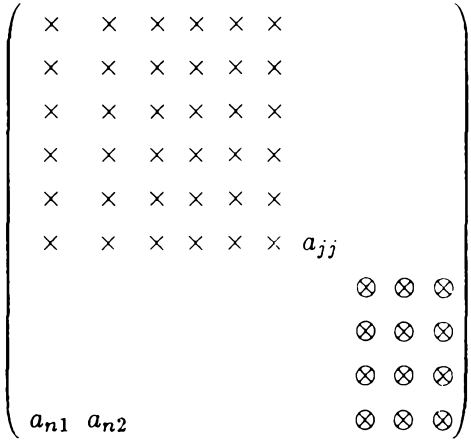
Solution. We note that a_{jj} has to be $j^2 - 1$ as there are $j^2 - 1$ entries to its top left. Also, there are $(n - j + 1)^2$ entries to its bottom right. Hence

$$a_{jj} \leq n^2 - (n - j + 1)^2 + 1.$$

This says that

$$a_{jj} \in \left\{j^2, j^2 + 1, j^2 + 2, \cdots, n^2 - (n - j + 1)^2 + 1\right\}$$

(see the matrix)



Therefore,

$$b_j \leq n^2 - (n-j+1)^2 + 2 = (2n-2)j - 2j^2 - (2n-1)$$

Therefore,

$$\sum b_j \leq (2n-2)\sum j - 2\sum j^2 - n(2n-1) = \frac{n}{3}(n^2 - 3n + 5)$$

SOLUTIONS COMPILED BY: PROF. K.N.RANGANATHAN, Professor of Mathematics, Sri RKM Vivekananda College, Chennai-600004.

READERS' COMMUNICATIONS

1. Prof.K.S.Padmanabhan,* (M73/2, 31^{st} Cross Street, Besant Nagar, Chennai-600090), Former Director of the Ramanujan Institute for Advanced Study in Mathematics, University of Madras, (Life member of AMTI) has given the following Alternate Solutions to some Problems and their Generalisations.

Problem 1 Vol. 35 (1999) p.230.
Five distinct points A, B, C, D and E lie on a line with $AB = BC = CD = DE$. The point F lies outside the line. Let G be the circumcentre of the triangle ADF and H the circumcentre of the triangle BEF. Show that the lines GH and FC are perpendicular.

Solution. The solution that has appeared uses analytical geometry. We give below a more general problem and solve it by vector methods.

We prove the following:

Theorem. $A_1, A_2, A_3, \cdots, A_{2n-1}$ are $(2n-1)$ points lying on a straight line such that $A_1A_2 = A_2A_3 = \cdots = A_{2n-2}A_{2n-1} = a$. S_1, S_2 are the circumcentres of $\triangle A_1FA_{2n-2}$, $\triangle A_2FA_{2n-1}$ respectively, where F is a point outside the line. Then S_1S_2 is perpendicular to FA_{n+1}.

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* We regret to report the demise of the author on 2 June 2002. Ed.

Proof.

$$\overline{FS_1}\cdot\overline{FS_1} = FS_1^2 = \left(\overline{A_{n+1}F} - \overline{A_{n+1}S_1}\right)^2$$
$$= A_{n+1}F^2 + A_{n+1}S_1^2 - 2\overline{A_{n+1}F}\cdot\overline{A_{n+1}S_1}\ . \qquad (1)$$

$$FS_2^2 = \left(\overline{A_{n+1}F} - \overline{A_{n+1}S_2}\right)^2$$
$$= A_{n+1}F^2 + A_{n+1}S_2^2 - 2\overline{A_{n+1}F}\cdot\overline{A_{n+1}S_2}\ . \qquad (2)$$

$$\therefore (FS_1^2 - A_{n+1}S_1^2) - (FS_2^2 - A_{n+1}S_2^2) = 2\overline{A_{n+1}F}\cdot\overline{S_1S_2} \qquad (3)$$

L.H.S. of (3) $= (S_1A_1^2 - A_{n+1}S_1^2) + (A_{n+1}S_2^2 - S_2A_{2n+1}^2)$, since $S_1F = S_1A; S_2F = S_2A_{2n+1}$. (4)

But $S_1A_1^2 - S_1A_{n+1}^2 = A_1L^2 - LA_{n+1}^2$ where L is the foot of the perpendicular on the given line from S. Also $AL = (n-\frac{1}{2})a$ and $LA_{n+1} = a/2$ so that $S_1A_1^2 - S_1A_{n+1}^2 = (n-1/2)^2a^2 - a^2/4LA_{n+1}^2$. Similarly

$$S_2A_{n+1}^2 - S_2A_{2n+1}^2 = A_{n+1}M^2 - MA_{2n+1}^2 = a^2/4 - (n-\frac{1}{2})^2a^2,$$

where M is the foot of the perpendicular from S_2, on the given line.

From (4),(5) and (6) we find L.H.S. of (3)=0, implying $\overline{A_{n+1}F}\cdot\overline{S_1S_2} = 0$. This proves that S_1S_2 is perpendicular to $A_{n+1}F$. (Q.E.D.)

Remark 1. For $n = 2$, we get the solution to problem 1 as a particular case.

Remark 2. We can also consider the case of an even number of points $A_1, A_2, \cdots, A_{2n}$ such that $A_1A_2 = A_2A_3 =$

$\cdots = A_{2n-1}A_{2n} = a$ and prove along similar lines that $S_1S_2 \perp CF$ where C is the mid-point of A_1A_{2n} and F lies outside the line through $A, A_2, \cdots, A_{2n}$, given that S_1, S_2 are the circumcentres of $\triangle A_1FA_{2n-1}, \triangle A_2FA_{2n}$ respectively.

REMARKS BY A REFEREE: The theorem stated as also the result suggested in Remark 2 above is no generalization of Problem 1, if Problem 1 is understood in the following sense: A, B, D, E are points on a line such that $AB = DE$ and C is the midpoint of AE. F is a point outside the line with G and H being circumcentres of ADF and BEF respectively. Then $GH \perp FC$.

Problem 2 Vol. 35 (1999) p.258.

If a, b, c are the sides of a triangle, prove that

$$\frac{a}{c+a-b} + \frac{b}{a+b-c} + \frac{c}{b+c-a} \geq 3.$$

We prove the following result, of which the above is a particular case:

Theorem 2. Let $a_1, a_2, \cdots, a_n$ be be positive quantities such that the sum of any $(n-1)$ of them is greater than $(n-2)$ times the remaining one. Then

$$a_1/(a_n + a_{n-1} + \cdots + a_3 + a_1 - (n-2)a_2) +$$
$$a_2/(a_n + \cdots + a_4 + a_2 + a_1 - (n-2)a_3) + \cdots +$$
$$a_n/(a_n + \cdots - (n-2)a_1) \geq n. \qquad \text{(A)}$$

Proof. Let $(n-1)s = a_1 + a_2 + \cdots + a_n$ so that

$$(n-1)(s-a_i) = \sum_{r=1}^{n} a_r - (n-1)a_i, \text{ for } i = 1, 2, \cdots, n.$$

The left side of the inequality to be proved can be written as

$$\frac{1}{(n-1)}\{a_1/(s-a_2) + a_2/(s-a_3) + \cdots + a_n/(s-a_1)\}$$

It suffices to prove that

$$\frac{a_1}{s-a_2} + \frac{a_2}{s-a_3} + \cdots + \frac{a_n}{s-a_1} \geq n(n-1). \qquad (5)$$

L.H.S. of (5)

$$\begin{aligned}
&= \frac{(n-1)s - (a_2 + a_3 + \cdots + a_n)}{(s-a_2)} + \cdots \\
&\quad + \frac{(n-1)s - (a_1 + a_2 + \cdots + a_{n-1})}{(s-a_1)} \\
&= \frac{(s-a_2) + (s-a_3) + \cdots + (s-a_n)}{(s-a_2)} + \cdots \\
&\quad + \frac{(s-a_1) + (s-a_2) + \cdots + (s-a_{n-1})}{(s-a_1)} \\
&= n + \{(s-a_3)/(s-a_4) + \cdots + (s-a_n)/(s-a_2)\} + \cdots \\
&\quad + \left\{\frac{s-a_2}{s-a_1} + \cdots + \frac{s-a_{n-1}}{s-a_1}\right\} \qquad (6)
\end{aligned}$$

There are $(n-2)$ fractions in each of the braces and there are n braces. Each of the quantities $(s-a_1), (s-a_2)\cdots,$ $(s-a_n)$ occurs $(n-2)$ times as the denominator of a fraction and each of them appears as a numerator of a fraction once in each of the $(n-2)$ braces. So the product of all the fractions is precisely unity. By using the inequality of the

arithmetic mean and geometric mean of positive quantities, we get

$$\frac{1}{n(n-2)}\left[\left\{\frac{s-a_3}{s-a_2}+\cdots+\frac{s-a_n}{s-a_2}\right\}+\{\ \}+\cdots\right.$$

$$\left.+\left\{\frac{s-a_2}{s-a_1}+\frac{s-a_3}{s-a_1}+\cdots+\frac{s-a_{n-1}}{s-a_1}\right\}\right]\geq 1.$$

Thus R.H.S. of (6) $\geq n+n(n-2)=n(n-1)$, proving the result.

Remark 3. For $n=3$, one gets the result in Problem 2.

Problem 3 Vol. 35 (1999) p.266.

Let a,b,c be three real numbers such that $1\geq a\geq b\geq c\geq 0$. Prove that if λ is a root of the cubic equation $x^3+ax^2+bx+c=0$, real or complex, then $|\lambda|\leq 1$.

The following more general result is proved in a simpler way.

Theorem 3. Let $a_1,a_2,\cdots,a_n$ be real numbers such that $1\geq a_1\geq a_2\geq a_3\geq\cdots\geq a_n\geq 0$. If λ is a root of the equation $x^n+a_1x^{n-1}+\cdots+a_n=0$, then $|\lambda|\leq 1$.

Proof. If $a_n=0, x=0$ is a root and on division by x we get an equation of degree $(n-1)$ for which we have to prove the assertion. So let $a_n>0$. Now $x=0$ is not a root. If $a_n>0$, the equation has no positive root. Consider for $x\neq 1$ and $|x|\geq 1$ the following

$$(x-1)(x^n + a_1x^{n-1} + \cdots + a_n)$$
$$= x^{n+1} - x^n(1-a_1) - x^{n-1}(a_1 - a_2) - \cdots - x(a_{n-1} - a_n) - a_n$$
$$= x^{n+1}\left\{1 - \left(\tfrac{1-a_1}{x} + \tfrac{a_1-a_2}{x^2} + \cdots + \tfrac{a_{n-1}-a_n}{x^n} + \tfrac{a_n}{x^{n+1}}\right)\right\}$$

$$\therefore \quad |(x-1)(x^n + a_1x^{n-1} + \cdots + a_n)|$$
$$\geq |x^{n+1}|\{1 - |(1-a_1)/x + (a_1-a_2)/x^2 + \cdots + a_n/x^{n+1}|\}$$
$$> |x^{n+1}|\{1 - [(1-a_1) + (a_1-a_2) + \cdots + (a_n)]\}.$$

Strict inequality holds in the last step, since $(1-a_1)/x, (a_1-a_2)/x^2, \cdots, (a_{n-1}-a_n)/x^n, a_n/x^{n+1}$ have different arguments except when x is real and positive, in which case the theorem is trivial. (Note that in the latter case $x^n > 0$ and the terms $a_1x^{n-1}, a_2x^{n-2}, \cdots a_n \geq 0$, so that $x^n + a_1x^{n-1} + \cdots + a_n \neq 0 (> 0)$). So, we have

$$\begin{aligned} & |(x-1)(x^n + a_1x^{n-1} + \cdots + a_n)| \\ > \quad & |x^{n+1}|\{1 - ((1 - a_1 + a_1 - a_2 + \cdots + a_n)\} \\ > \quad & 1 - (1 - a_1 + a_1 - a_2 + \cdots + a_n) \\ = \quad & 0. \end{aligned}$$

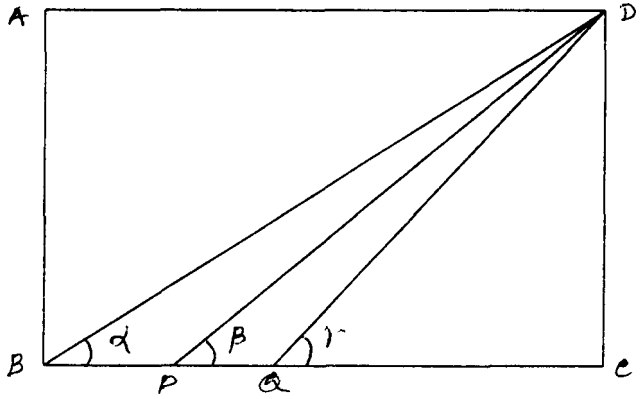
which is impossible if x were a root of the given equation which is greater than or equal to unity. This shows that all the roots of the given equation have absolute value less than unity.

Remark. We have actually proved a result stronger than required in the form $|\lambda| \lneqq 1$.

2. Dr. S.K. Srivatsa, Professor of Electronics and Communication Engineering, Anna University, Chennai (Life member of AMTI) has given the following alternative easy solution to **Problem No.2 of Vol. 36 (2000) p.141.**

Problem. Let $ABCD$ be a rectangle with $BC = 3AB$. Show that if P, Q are the points on BC with $BP = PQ = QC$, then $\angle DBC + \angle DPC = \angle DQC$.

Solution. The above mentioned problem can be solved in a very simple way using Trigonometry as shown below:



From $\triangle DQC, \tan\gamma = \frac{DC}{QC} = \frac{\frac{1}{3}BC}{\frac{1}{3}BC} = 1$ whence $\gamma = 45°$.

From $\triangle DBC, \tan\alpha = \frac{DC}{BC} = \frac{1}{3}$

and from $\triangle DPC, \tan\beta = \frac{DC}{PC} = \frac{\frac{1}{3}BC}{\frac{2}{3}BC} = \frac{1}{2}$

Also $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{6}} = 1$

whence $\alpha + \beta = 45°$

$\therefore\ \angle DBC + \angle DPC = \angle DQC$ Q.E.D.

INTERNATIONAL HONOUR FOR IIT KANPUR

The international scientific community was truly stunned when the New York Times reported on a new discovery from IIT Kanpur.

The alacrity with which the Americans woke up to the the discovery along with its far reaching implications in web related technologies surprised many in the US.

On the face of it, the discovery sounds simple: how to find out whether a number is prime or a composite number. This is easy to solve for small numbers but not for very large numbers. Mathematicians from all over the world have been working to solve this for more than a few hundred years. This problem had also haunted theoretical computer scientists.

Now, a truly world-class theoretical computer science achievement has come out from India solving this.

The "primality" algorithm of Manindra Agrawal, Neeraj Kayal and Nitin Saxena of IIT Kanpur, released earlier this month, is the answer to these worries.

Given a number, this algorithm checks whether it is composite (can be divided by a smaller number, a "factor") or prime (the only smaller number dividing it is 1).

The IIT 'algorithm' works better for large numbers than previous methods. Agrawal is a researcher in his thirties, well known for his work in the theoretical computer science community. Kayal and Saxena are in their twenties, and part

of this work was done while they were doing their B.Tech. in computer science.

The primality algorithm just declares a number to be composite but doesn't yield the smaller number dividing it. If the method can be extended to also giving a smaller divisor (a "factorization algorithm"), computer programs will be able to crack security codes–for example–of the kind used in credit cards. So the whole e-commerce industries will have to operate with new security algorithms.

There have been excellent theoretical computer science results by Indians before this, but many of them have been working abroad. One of them, Madhu Sudan of MIT, has just been awarded the Nevanlinna prize, a top award in theoretical computer science presented at the International Congress of Mathematicians (recently held in Beijing).

One of the many results he obtained was a better algorithm for checking, given a mathematical proof, whether it is correct or not. The algorithm, developed with four other authors, works most of the time but has a small probability of failure. This means that by running the same program several times, the chances of a mistake become lower.

An algorithm of this kind was developed for the primality problem by Michael Rabin many years ago, and the importance of the IIT Kanpur result is that it always works, with no chance of failure.

Courtesy: K. VENKATESH, Sify News

NUMBER THEORY HITS 'PRIME' TIME

The *New York Times* has reported it, and so have the local Indian newspapers. The internet is abuzz with news about Mahindra Agrawal, Neeraj Kayal and Nitin Saxena, mathematicians working in the Indian Institute of Technology, Kanpur, who have come up with something of a breakthrough in methods to deal with very large prime numbers.

Primes are simply numbers that do not have divisors. The numbers 3,5,7, for instance, have no divisors and these are prime numbers. But as the number 9, can be divided by 3, it is not a prime. We can, of course, easily see that primes just have to be odd numbers, because even numbers are all divisible by 2.

On the face of it this may seem to be, well, just one feature of a number, which makes things a little more difficult if it tunned up in a fraction, and nothing more. But, in fact, prime numbers are star players in a fascinating pantomime of sleights and computing stratagems in 'pure' mathematics, which held such magic for mathematical greats who brought to all scientists the tools with which to access the secrets of the world over the centuries.

This magic comes alive as soon as we venture beyond the first few hundred primes, which can be easily identified as primes, by trying a good many divisions. As we reach numbers in tens of thousands - which are small change for computers - identifying or producing a prime gets

devilishly difficult and generations of mathematicians have spent lifetimes in devising clever and quicker ways. And now, the age of computers and e-commerce has discovered a straight commercial worth of doing just this in this in the quickest way!

A major concern while buying and selling on the internet is that all sides need to be sure of the authenticity and credibility of the messages exchanged. One way of making sure is by agreeing on a secret code every time we transact on the Net. One trouble with this is that we can hardly arrange, every time, to first meet the other person and agree on the code. But what is more important is that even if we did, most codes could be 'cracked'. And computers have the persistence and speed to make most codes fairly 'easy'!

A way out was shown by the devilish difficulty of generating or identifying huge primes. A code is generated using the result of multiplying a secret prime number of the sender with another secret prime number of the receiver. The code is such that it could be deciphered by either secret number and by nothing else. Cracking the code, in fact, consists simply of finding the only two numbers that go into the product.

The security of the system is that if the two primes chosen are of the order of 2 to the square of 100 - or '2' multiplied by itself a hundred times - which is a number with 33 digits, the product is too large to be easily broken into its two factors! The product would typically be around 2 to the square of 200, which is a number of around 66 digits. Even if only one

in a million of all the possible factors that one could have with 33 digits were checked, to find divisors, the number of divisions to be done would have 27 digits! If a computer could do a million divisions a second, the number of seconds it would take would still have 21 digits. It is not difficult to work out that this would be thousands of years! In fact, the methods used are 'smarter', like trying to divide only by primes, but the time taken is still of the same order.

It is in this context that the study of numbers, and particularly of primes, is of such importance today. The method the three young mathematicians of IIT, Kanpur, have developed has been hailed as a solution to one of the great, unsolved problems in theoretical computer science and computational number theory. "It's the best result I've heard of in over 10 years", says Shafi Goldwasser, a professor of computer science at the Massachusetts Institute of Technology and the Weizmann Institute of Science in Israel.

By S. ANANTHANARAYANAN
The writer is a civil servant working in Mumbai
Courtesy: *The Hindu*

PROF. K.S. PADMANABHAN PASSES AWAY

Professor K.S. Padmanabhan (known as KSP amongst his colleagues, friends and students) passed away on 2^{nd} June 2002 at Chennai leaving his family and friends in great grief. Born on 19-03-1930 at Palayamkottai, he studied upto school final at Ambasamudram and later had his graduation from St. Xavier's College, Palayamkottai. He took his M.A. degree in Mathematics from St. Joseph's College, Tirichirappali in 1951. After serving for a short period as Lecturer in the SRR & CVR College, Vijayawada, he joined Annamalai University where he served till 1969 in the capacity of Lecturer and later Reader. There he obtained M.Sc. degree and Ph.D. degree under the guidance of Professor V. Ganapathy Iyer. He then shifted to University of Madras as Reader in the Ramanujan Institute for Advanced Study in Mathematics from where he retired in June 1990 as Director and Head.

He was a famous teacher and an ardent researcher. He has published a number of papers on univalent functions, star-like functions, quasi-conformal mappings, Riemann surfaces and analytic function theory in national and international journals. He was for many years the Librarian of Indian Mathematical Society. He was also a founder member

to Mathematics. Even after his retirement he was associated as a guest faculty with Pondicherry University and Calicut University. He had guided a number of students in doctoral research.

The Ramanujan Mathematical Society has launched on a fund collection with a view to institute an endowment in memory of Prof. KSP and contributions to this fund are solicited from all his students, friends and lovers of Mathematics. Contributions which are exempt from Income-tax u/s 80(G) (order of Commissioner of Income Tax, Tirichirappalli C.No. 6162 E(10)/2002-2003 dated 9-7-2002) may kindly be sent by Cheque / DD in favour of *Professor K.S. Padmanabhan Memorial Fund* to Professor P. Paulraja, Treasurer, Ramanujan Mathematical Society, Department of Mathematics, Annamalai University, Annamalai Nagar-608002 so as to reach him not later than 15-11-2002.

On behalf of AMTI, we offer our heartfelt condolences to the members of the bereaved family and pray to the Almighty that the departed soul may rest in peace.

We take it as a privilege to publish, perhaps, his last paper communicated in the month of May, 2002 to the Editor, Mathematics Teacher elsewhere in this issue.

PROF. A. RAMAKRISHNAN,
Editor, Mathematics Teacher.

The Association of Mathematics Teachers of India

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